GEOMETRICAL DEDUCTIONS

BOOK I.

BLAIKIE AND THOMSON

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GEOMETRICAL DEDUCTIONS

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A TEXT-BOOK OF

GEOMETRICAL DEDUCTIONS

BOOK I.

Corresponding to Euclid, Book I.

BY

(Andrew)

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PREFACE

The object of this treatise is to afford a systematic course of training in the art of solving Geometrical Deductions or Riders. With this view it is divided into sections, each section consisting of three parts. There is first a deduction worked out in full, which is intended to serve as a model for the student. This is followed by a number of similar deductions, which are to be written out by the student, the figure being given in each case, and such hints regarding the mode of solution as experience shows are required by beginners. Lastly, each section contains some deductions to be accomplished without this aid, no figures or assistance being given except an occasional reference to the proposition on which the proof depends, or to a previous example.

As a rule it is desirable that the proofs should depend upon propositions of Euclid, and not upon previous examples, the only exception being in the case of certain standard theorems which are indicated in the text.

For convenience of reference, especially in the case of those who have used text-books other than Euclid's, the

enunciations of Euclid's propositions are given in an Appendix.

It is not necessary, and perhaps not desirable, that on his first reading the student should work through every example in each section. He should in each case, however, write out a sufficient number to insure his mastery of the principles involved; the others will be found useful when he comes to revise.

Through the kindness of friends the book has been tested, when in proof, by actual work with pupils, and the satisfactory result of this experiment has encouraged the authors to believe that the treatise may be found generally useful.

They have to acknowledge valuable suggestions and assistance from Messrs. Butters, Clark, and Walker, Heriot's Hospital School, Edinburgh; Mr. R. F. Davis; the Rev. W. F. Failes, Westminster School; Mr. Hayward, Harrow School; Mr. Macdonald, Daniel Stewart's College, Edinburgh; Dr. Mackay, Edinburgh Academy; Rev. J. J. Milne; Dr. Muir, Glasgow High School; Professor Raitt, Glasgow Technical College; Mr. Robertson, Edinburgh Ladies' College; Rev. G. Style, Giggleswick School; Mr. Tucker, University College School, and other friends.

Additional parts, corresponding to the remaining books of Euclid, are in preparation.

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DEFINITIONS.

SYMBOLS.

$\therefore$	signifies	*therefore.*
$=$	„	*is equal to.*
$\equiv$	„	*is equal in every respect to, is congruent with,* or *is identically equal to.*

Congruent figures are such as can be made to coincide by superposition.

$\perp$	signifies	*is at right angles to* or *is perpendicular to.*
$\parallel$	„	*is parallel to.*
$>$	„	*is greater than.*
$<$	„	*is less than.*
$+$	„	*together with.*
$-$	„	*diminished by.*
$\triangle$	„	*triangle.*
$\angle$	„	*angle.*
rt. $\angle$	„	*right angle.*
▱	„	*parallelogram.*
AB^2	„	*square on* AB.
$AB \cdot CD$	„	*rectangle contained by* AB *and* CD.
$AB \sim CD$	„	*the difference between* AB *and* CD.

CHAPTER I.

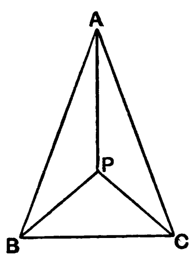
THEOREMS.

§ 1. (*Bookwork*, EUCLID, I. 1-4.)

1. Any point on the bisector of the vertical angle of an isosceles triangle is equally distant from the extremities of the base.

Let ABC be an isosceles triangle, and let AP bisect its vertical angle; it is required to prove that

$$BP = CP.$$



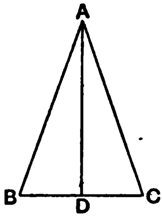
In $\triangle$s BAP, CAP $\left\{ \begin{array}{l} BA = CA, \\ AP = AP, \\ \angle BAP = \angle CAP; \end{array} \right.$ [Hypothesis. ... [Hypothesis.

$\therefore \triangle BAP \equiv \triangle CAP$; [Euc. I. 4.

$\therefore BP = CP.$

2. The straight line which bisects the vertical angle of an isosceles triangle bisects the base, and is at right angles to the base.

(*A Standard Theorem.*)



Show, as in above proof, that

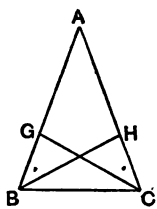
$$\triangle BAD \equiv \triangle CAD,$$

and hence

(1) $BD = CD$,

(2) $\angle ADB = \angle ADC =$ rt. $\angle$. [Def. of rt. $\angle$.

3. ABC is an isosceles triangle, and in AB, AC points G, H are taken so that AG = AH. Show that (1) BH = CG, (2) $\angle$ ABH = $\angle$ ACG, (3) $\angle$ AHB = $\angle$ AGC.



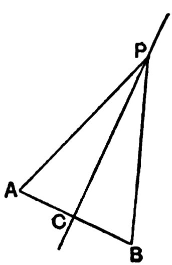
Use Euc. I. 4 to show that

$$\triangle BAH \equiv \triangle CAG.$$

Examine the case in which G and H are on AB and AC produced.

4. If through the mid-point of a given straight line a perpendicular to the line be drawn, any point in the perpendicular shall be equally distant from the extremities of the given line.

(*A Standard Theorem.*)



Use Euc. I. 4 to show that

$$\triangle ACP \equiv \triangle BCP.$$

NOTE.—*Standard Theorems* may be used in proving other deductions in the same way as Euclid's propositions, but instead of referring to them by number, the student should quote the enunciation.

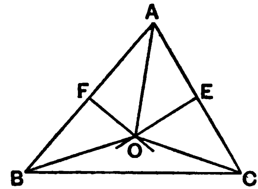
5. If through the mid-points of two sides of a triangle perpendiculars to these sides be drawn, their point of intersection shall be equally distant from the three angular points of the triangle.

Show as in Ex. 4 that

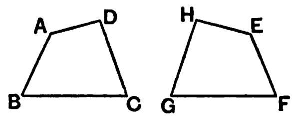
(1) AO=BO,

(2) AO=CO.

Observe that the point O need not be within the triangle.



6. If two quadrilateral figures have three sides and the two contained angles of the one respectively equal to the corresponding sides and angles of the other, they shall be congruent, *i.e.* equal in every respect.



Let AB=EF, BC=FG, CD=GH,

$\angle$ B= $\angle$ F, $\angle$ C= $\angle$ G.

Use superposition as in Euc. I. 4.

7. If two quadrilaterals have three angles and the two included sides of the one respectively equal to the corresponding angles and sides of the other, they shall be equal in all respects.

Use superposition.

8. If AB, or AB produced, bisect CD at right angles, then $\triangle$ ACB $\equiv$ $\triangle$ ADB.

Use Euc. I. 4.

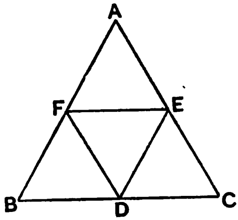
9. The $\triangle$ ABC is folded over the side AC so as to come into the position AB′C; show that BB′ $\perp$ AC.

Use Euc. I. 4.

10. AB is a finite straight line, and AC, AD make equal angles with it on opposite sides. If AC=AD, show that BC, BD also make equal angles with AB.

§ 2. (*Bookwork*, EUCLID, I. 1-6.)

1. The triangle formed by joining the mid-points of the sides of an equilateral triangle is itself equilateral.



Let D, E, F be the mid-points of the sides BC, CA, AB, of the equilateral △ ABC; it is required to prove that

△ DEF is equilateral.

In △s FAE, FBD $\begin{cases} FA = FB, & \text{[Hypothesis.} \\ AE = BD, & \text{[Halves of equal lines.} \\ \angle A = \angle B; & \text{[Euc. I. 5.} \end{cases}$

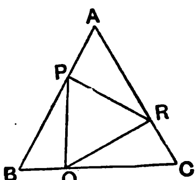
∴ △ FAE ≡ △ FBD; [Euc. I. 4.

∴ FE = FD.

Similarly it can be proved that FD = DE;

∴ △ DEF is equilateral.

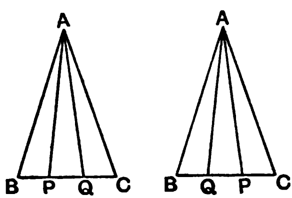
2. P, Q, R are points in the sides AB, BC, CA of an equilateral triangle such that AP = BQ = CR, show that △ PQR is equilateral.



Use Euc. I. 5 and I. 4 to show that

△ PAR ≡ △ QBP ≡ △ RCQ.

3. P and Q are points in the base of an isosceles triangle, such that BP = CQ; show that AP = AQ.

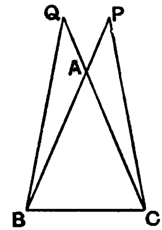


Use Euc. I. 5 and I. 4 to prove

△ ABP ≡ △ ACQ.

Examine the cases in which P and Q are on BC produced.

4. The equal sides BA, CA of an isosceles triangle are produced beyond the vertex to P and Q, so that AP=AQ; show that BQ=CP.



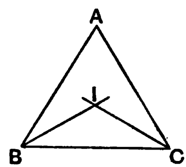
Prove BP=CQ.

Use Euc. I. 5 and I. 4 to show

$\triangle$ PBC$\equiv$$\triangle$ QCB.

Compare § 1, Ex. 3.

5. The bisectors of the angles at the base of an isosceles triangle are produced to meet; show that the triangle thus formed is isosceles.

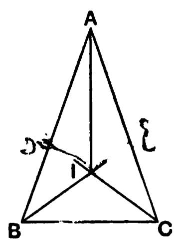


Use Euc. I. 5 and Axiom 7 to show

$\angle$ IBC$=$$\angle$ ICB.

Use Euc. I. 6 to show BI=CI.

6. The bisectors of the three angles of an isosceles triangle meet in a point.



Let the bisectors of $\angle$s B and C meet in I. Join AI.

Show, as in Ex. 5, that BI=CI.

Use Euc. I. 4 to prove $\triangle$ ABI$\equiv$$\triangle$ ACI; and hence show that AI is the bisector of $\angle$ A.

7. From AB, AC, the equal sides of an isosceles triangle, equal lengths AD, AE are cut off. BE and CD meet in P. Show that AP bisects $\angle$A.

Use Euc. I. 4 and I. 6.

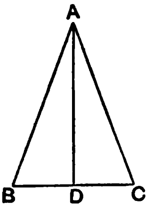
8. The straight lines joining the angular points of an isosceles triangle to the mid-points of the opposite sides meet in a point.

Use the methods of the previous Ex. and § 1, Ex. 2.

§ 3. (*Bookwork*, Euclid, I. 1-8.)

1. The straight line which joins the vertex of an isosceles triangle to the mid-point of the base, bisects the vertical angle and is perpendicular to the base.

Let ABC be an isosceles triangle having AB = AC, and let BD = CD; it is required to prove (*A Standard Theorem.*)



(1) $\angle$ BAD = $\angle$ CAD,

(2) AD $\perp$ BC.

In $\triangle$s ABD, ACD $\begin{cases} AD = AD, \\ AB = AC, & \text{[Hypothesis.} \\ BD = CD; & \text{[Hypothesis.} \end{cases}$

$\therefore$ $\triangle$ ABD $\equiv$ $\triangle$ ACD; [Euc. I. 8.

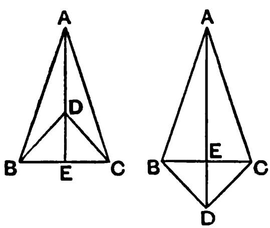
$\therefore$ (1) $\angle$ BAD = $\angle$ CAD,

(2) $\angle$ ADB = $\angle$ ADC = a rt. $\angle$.

[Def. of rt. $\angle$.

Observe that this proposition is a converse to § 1, Ex. 2. It may be proved by Euc. I. 4 and I. 5 without employing Euc. I. 8.

2. The straight line which joins the vertices of two isosceles triangles on the same base, bisects the base and is perpendicular to it.



Use Euc. I. 8 to show that

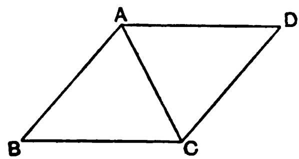
$\triangle$ ABD $\equiv$ $\triangle$ ACD,

and $\therefore$ $\angle$ BAD = $\angle$ CAD.

Compare $\triangle$s BAE, CAE.

Note.—A quadrilateral which is divided by one diagonal into two isosceles triangles is sometimes called a *kite*. The second part of the above theorem may therefore be thus expressed: *The diagonals of a kite intersect at right angles.* Observe that the diagonal which joins the vertices of the isosceles triangles divides the kite into two congruent triangles, and that it also bisects the other diagonal.

3. The opposite angles of a rhombus are equal, and the diagonals bisect them.



ABCD is a rhombus;

that is, AB = BC = CD = DA.

Draw the diagonal AC.

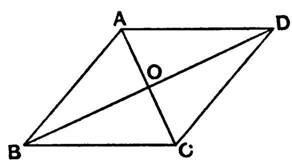
Use Euc. I. 8 to show that △ ABC ≡ △ ADC.

4. The diagonals of a rhombus bisect each other, and are at right angles to each other.

Show, as in Ex. 3, that

∠ BAC = ∠ DAC.

Compare △s BAO, DAO.



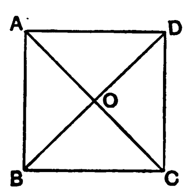
Observe that this is a particular case of Ex. 2.

5. The diagonals of a square (1) are equal to each other, (2) bisect each other, (3) are at right angles to each other, and (4) bisect the angles of the square.

For (1) show by Euc. I. 4 that

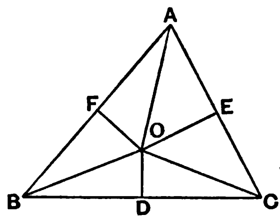
△ BAD ≡ △ ABC.

For (2), (3), and (4) use the methods of Exx. 3 and 4.



6. The straight lines drawn at right angles to the sides of a triangle from their mid-points, meet in a point which is equally distant from the three vertices of the triangle.

(*A Standard Theorem.*)



Let D, E, F be the mid-points of BC, CA, AB.
Draw EO, FO $\perp$ CA, AB. Join OD.
As in § 1, Ex. 5, show that
AO = BO = CO.
Use Euc. I. 8 to prove $\triangle$ BOD $\equiv \triangle$ COD,
and hence show OD $\perp$ BC.

Observe that the point O need not be within the triangle.

DEFINITION.—*Three or more straight lines which meet in a point are said to be* **concurrent.**

Thus Ex. 6 shows that the straight lines drawn at right angles to the sides of a triangle from their mid-points are concurrent.

7. If the opposite sides of a quadrilateral be equal, the opposite angles shall also be equal.

Use Euc. I. 8.

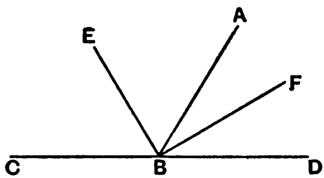
8. The straight line which joins the vertices of two isosceles triangles on the same base bisects both vertical angles.

9. ABCD is a quadrilateral in which the side AB = CD, and the diagonal AC = BD. Show that $\angle$ A = $\angle$ D, $\angle$ B = $\angle$ C, and that, if AC and BD intersect in O, the $\triangle$s OAD, OBC are isosceles.

§ 4. (*Bookwork*, Euclid, I. 1-14.)

1. The bisectors of the adjacent angles, formed by one straight line standing on another straight line, are at right angles to each other.

(*A Standard Theorem.*)



Let AB stand on CD and let BE, BF bisect the angles ABC, ABD; it is required to prove that

$\angle$ EBF is a right angle.

Because $\angle$ ABC is bisected

$$\angle ABE = \tfrac{1}{2} \angle ABC;$$

similarly $\angle ABF = \tfrac{1}{2} \angle ABD$;

$\therefore$ by addition $\angle EBF = \tfrac{1}{2} (\angle ABC + \angle ABD)$.

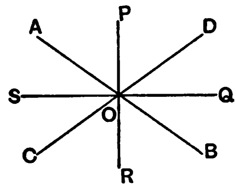
But $\angle ABC + \angle ABD = 2$ rt. $\angle$s; [Euc. I. 13.

$\therefore$ $\angle$ EBF is a right angle.

Definitions.—*Two angles which together make up a right angle are said to be* **complementary** *angles, and either angle is called the* **complement** *of the other. Two angles which together make up two right angles are said to be* **supplementary** *angles, and either angle is called the* **supplement** *of the other.*

Thus in Ex. 1, the $\angle$s ABE, ABF are complementary, and the $\angle$s ABC, ABD are supplementary.

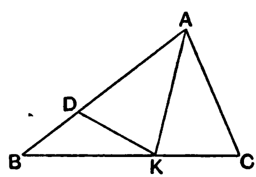
2. The bisectors of the four angles, which two intersecting straight lines make with each other, form two straight lines which are perpendicular to each other.



Let AB, CD be the given straight lines, etc.

Show as in Ex. 1 that $\angle$s POQ, QOR, etc., are right angles, and apply Euc. I. 14.

3. ABC is a triangle in which AB > AC. The bisector of the $\angle$ A meets the base in K. Show that $\angle$ AKB is obtuse.



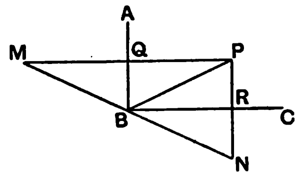
From AB (the greater) cut off AD = AC.

Use Euc. I. 4 to show that $\angle$ AKD = $\angle$ AKC.

Deduce $\angle$ AKB > $\angle$ AKC, and use Euc. I. 13.

4. If from a point within a right angle perpendiculars be drawn to the lines containing the angle, and each perpendicular be produced its own length, the extremities of the produced lines and the vertex of the right angle shall be in one straight line.

Let ABC be the given rt. ∠.
Use Euc. I. 4 to show that
∠ PBQ = ∠ MBQ, etc.
Hence show that
∠ PBM + ∠ PBN = 2 rt. ∠ s,
and apply Euc. I. 14.

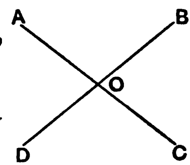


DEFINITION.—*Three or more points which are in the same straight line are said to be* **collinear.**

Thus, in Ex. 4, M, B, N are shown to be collinear.

5. Four straight lines, OA, OB, OC, OD, meet in a point making ∠ AOD = ∠ BOC, and ∠ AOB = ∠ COD. Prove that AO, OC (and BO, OD) are in the same straight line.

Use Euc. I. 13, Cor. ii.,* to show ∠ s AOB, BOC supplementary.



* This corollary is sometimes given as a corollary to Euc. I. 15.

6. P is a point within the △ ABC. Show that of the ∠s BPC, CPA, APB, at least two must be obtuse.

Use Euc. I. 13, Cor. ii.

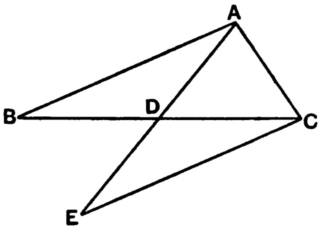
7. Prove § 2, Ex. 2, for the case in which P, Q, R are on the sides produced.

8. The corner of the leaf of a book is folded down. Show that the bisector of the angles formed by the edges of the leaf and the edges of the folded part are at right angles to the crease.

§ 5. (*Bookwork*, EUCLID, I. 1-20.)

1. The straight line drawn from any angular point of a triangle to the mid-point of the opposite side is less than half the sum of the other two sides.

(*A Standard Theorem.*)



In $\triangle$ ABC, let D be the mid-point of BC; it is required to prove that

$$AD < \tfrac{1}{2}(AB + AC).$$

Produce AD to E, so that DE = AD. Join EC.

In $\triangle$s ADB, EDC $\left\{\begin{array}{l} AD = ED, \\ DB = DC, \\ \angle ADB = \angle EDC; \end{array}\right.$ [Construction. [Hypothesis. [Euc. I. 15.

$\therefore \triangle ADB \equiv \triangle EDC$; [Euc. I. 4.

$\therefore AB = CE.$

But $AC + CE > AE$; [Euc. I. 20.

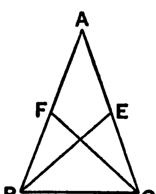
$\therefore AC + AB > 2AD,$

or $AD < \tfrac{1}{2}(AB + AC).$

DEFINITION. — *The straight line drawn from any vertex of a triangle to the mid-point of the opposite side is called the* **Median** *from that vertex.*

Thus Ex. 1 shows that the *median* drawn from any vertex is less than half the sum of the sides containing the angle.

2. In an isosceles triangle the medians drawn from the extremities of the base are equal.



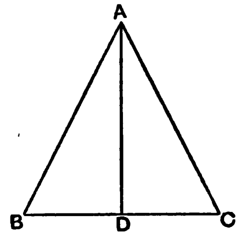
Use Euc. I. 4 to show that

$$\triangle BAE \equiv \triangle CAF.$$

3. If in any triangle a median be perpendicular to the side to which it is drawn, the triangle shall be isosceles.

Use Euc. I. 4 to show that

$$\triangle \mathrm{ADB} \equiv \triangle \mathrm{ADC}.$$



DEFINITION.—*The straight line drawn from a vertex of a triangle perpendicular to the opposite side, and terminated by that side, is called the* **Altitude** *from that vertex.*

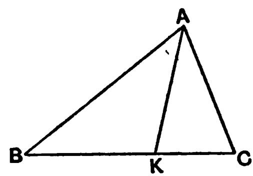
Thus in Ex. 3, AD is both median and *altitude.*

4. AK, the bisector of the $\angle$ A of the triangle ABC, meets BC in K; show that AB > BK and AC > CK. Hence obtain a second proof of Euc. I. 20.

Use Euc. I. 16 to show that

$\angle \mathrm{AKB} > \angle \mathrm{KAC}$ (or $\angle$ KAB),

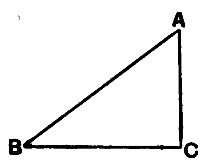
and apply Euc. I. 19.



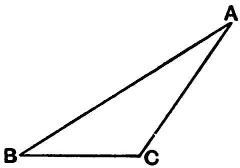
DEFINITION.—*In a right-angled triangle the side opposite to the right angle is called the* **Hypotenuse.**

5. In any right-angled triangle, the hypotenuse is greater than either of the sides containing the right angle.

Use Euc. I. 17 and I. 19.

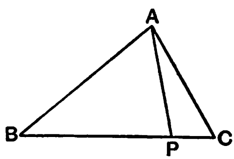


6. In any obtuse-angled triangle the side opposite to the obtuse angle is greater than either of the sides containing the obtuse angle.



Use Euc. I. 17 and I. 19.

7. ABC is a triangle in which AB > AC and P is any point in BC. Show that AB > AP.

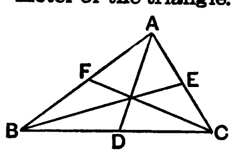


Use Euc. I. 18 to show that
$\angle ACB > \angle ABC$,
and Euc. I. 16 to show that
$\angle APB > \angle ACB$, etc.,
and apply Euc. I. 19.

DEFINITION.—*The sum of the sides of any rectilineal figure is called its* **Perimeter**.

Thus $BC+CA+AB$ is the *perimeter* of the $\triangle$ ABC.

8. The sum of the medians of a triangle is less than the perimeter of the triangle.



Use Ex. 1 for the three medians in turn, add the results, and divide by 2.
(See note on § 1, Ex. 4, as to the use of standard theorems.)

9. Of the three exterior angles of a triangle, any two are together greater than two right angles.

Use Euc. I. 17.

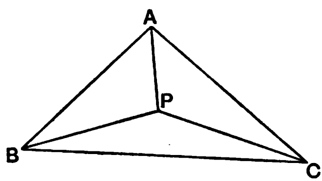
10. In the quadrilateral ABCD, AB is the longest and CD the shortest side. Show that $\angle$ C > $\angle$ A and $\angle$ D > $\angle$ B.

§ 6. (*Bookwork*, Euclid, I. 1-21.)

1. If any point be taken within a triangle, the sum of its distances from the angular points shall be (1) greater than half the perimeter of the triangle, and (2) less than the perimeter.

Let P be any point within $\triangle$ ABC; it is required to prove that

$$\text{(1)}\quad PA+PB+PC>\tfrac{1}{2}(BC+CA+AB),$$
$$\text{(2)}\quad PA+PB+PC<BC+CA+AB.$$



(1) In $\triangle$ PBC, $PB+PC>BC$. [Euc. I. 20.

Similarly, $PC+PA>CA$,

$PA+PB>AB$;

$\therefore$ by addition,

$$2(PA+PB+PC)>BC+CA+AB;$$
$$\therefore\ PA+PB+PC>\tfrac{1}{2}(BC+CA+AB).$$

(2) In $\triangle$ ABC, $CA+AB>PB+PC$, [Euc. I 21.

$AB+BC>PC+PA$,

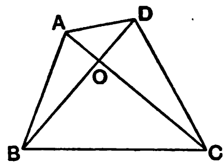
$BC+CA>PA+PB$;

$\therefore$ by addition,

$$2(BC+CA+AB)>2(PA+PB+PC);$$
$$\therefore\ BC+CA+AB>PA+PB+PC,$$

or $PA+PB+PC<BC+CA+AB$.

2. The perimeter of a quadrilateral is (1) greater than the sum of the diagonals, and (2) less than twice the sum of the diagonals.



For (1) use Euc. I. 20 to prove that

$$AB + BC > AC, \quad BC + CD > BD, \text{ etc.};$$

$$\therefore \; 2(AB + BC + CD + DA) > 2(AC + BD), \text{ etc.}$$

For (2) use Euc. I. 20 to show that

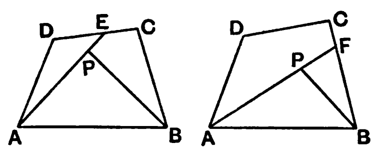
$$OA + OB > AB, \text{ etc.}$$

3. Any three sides of a quadrilateral are together greater than the fourth side.

Draw a diagonal and use Euc. I. 20.

4. If P be a point within a quadrilateral ABCD,

$$BC + CD + DA > AP + PB.$$



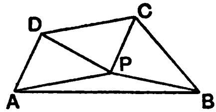
Produce AP to meet BC or CD, and use Euc. I. 20 and the method of Ex. 3 to show in fig. 1 that

$$BC + CD + DA > AE + EC + CB > AP + PB,$$

and in fig. 2 that

$$BC + CD + DA > AF + FB, \text{ etc.}$$

5. If a point be taken within a quadrilateral, the sum of its distances from the angular points shall be (1) greater than one-half, and (2) less than three-halves of the perimeter.



For (1) use Euc. I. 20.

$$PA+PB>AB, \text{ etc.}$$

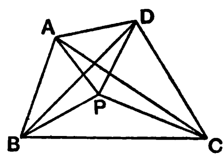
For (2) show, as in Ex. 4, that

$$BC+CD+DA>PA+PB, \text{ etc.}$$

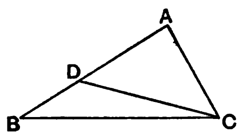
6. If within a quadrilateral ABCD, P be a point which is not the point of intersection of the diagonals,

$$PA+PB+PC+PD>AC+BD.$$

Use Euc. I. 20.



7. The difference between two sides of a triangle is less than the third side.



From AB cut off $AD=AC$. Join CD.

Use Euc. I. 16 to show that

$$\angle BDC>\angle ACD \text{ and } \angle ADC>\angle BCD.$$

Apply Euc. I. 5 and I. 19 to deduce $BC>BD$.

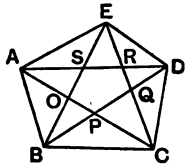
Otherwise. $AC+BC>AB$, [Euc. I. 20.

or $AC+BC>AD+BD$.

But $AC=AD$;

$\therefore BC>BD$.

8. The perimeter of a pentagon is greater than one-half the sum of the diagonals and is less than the sum of the diagonals.



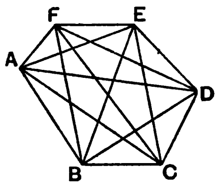
Use Euc. I. 20.

(1) $AB+BC>AC$, etc.,

(2) $AO+BO>AB$, etc.,

also $AC>AO+PC$, etc.

9. The perimeter of a hexagon is greater than one-half the sum of the diagonals which join alternate angles, and is also greater than two-thirds the sum of the diagonals which join opposite angles.



(1) $AB+BC>AC$, etc.

(2) $AB+BC+CD>AD$, etc.

10. All the sides but one of any polygon are together greater than that one.

Use Euc. I. 20.

11. Show that in any quadrilateral the sum of any two sides is greater than the difference of the other two.

Use the method of Ex. 10 or of Ex. 7.

12. If one triangle be wholly enclosed in another, the perimeter of the external triangle shall be greater than that of the internal.

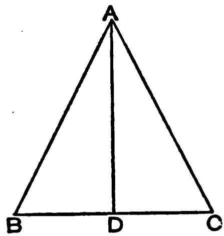
Produce the sides of the internal triangle in order, and use the method of Ex. 10.

13. The sum of the distances of any point from the angular points of a polygon is greater than half the perimeter of the polygon.

14. ABC is a triangle having $AB>AC$, and D is a point in BC such that $AD>AC$. From DA, DE is cut off $=AC$, and AE is bisected in F. If P be a point in FA, prove that $BP+PD>BA+AC$.

§ 7. (*Bookwork*, EUCLID, I. 1-26.)

1. In an isosceles triangle the altitude drawn from the vertex is also the median and the bisector of the vertical angle.



Let ABC be an isosceles triangle having AB=AC, and let AD be $\perp$ BC; it is required to prove that

$$(1)\quad BD = CD,$$
$$(2)\quad \angle BAD = \angle CAD.$$

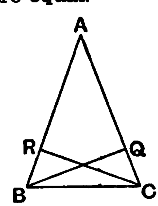
In $\triangle$s ABD, ACD $\begin{cases} AB = AC, & \text{[Hypothesis.} \\ \angle ABD = \angle ACD, & \text{[Euc. I. 5.} \\ \angle ADB = \angle ADC; & \text{[Right angles.} \end{cases}$

$\therefore \triangle ABD \equiv \triangle ACD;$ [Euc. I. 26.

$\therefore (1)\ BD = CD;\quad (2)\ \angle BAD = \angle CAD.$

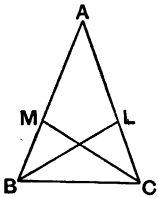
2. In an isosceles triangle the altitudes drawn from the extremities of the base are equal.

Use Euc. I. 26 to show $\triangle ABQ \equiv \triangle ACR$.



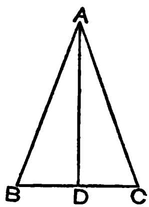
DEFINITION.—*The straight line drawn from a vertex of a triangle, bisecting the angle at that vertex and terminated by the opposite side, is called the* **Bisector** *of that angle.*

3. **In an isosceles triangle the bisectors of the angles at the base are equal.**



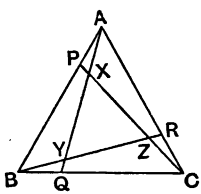
Use Euc. I. 5 and I. 26 to show either that $\triangle$ ABL $\equiv$ $\triangle$ ACM, or $\triangle$ MBC $\equiv$ $\triangle$ LCB.

4. **If in a triangle an altitude be also the bisector of the angle from which it is drawn, the triangle shall be isosceles.**



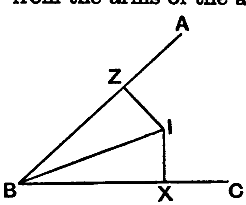
Use Euc. I. 26 to show that $\triangle$ ABD $\equiv$ $\triangle$ ACD.

5. **P, Q, and R are points in the sides of an equilateral $\triangle$ ABC, such that AP = BQ = CR. Show that the triangle formed by the intersection of AQ, BR, CP is equilateral.**



Use Euc. I. 4 to show that
$\triangle$ ABQ $\equiv$ $\triangle$ BCR $\equiv$ $\triangle$ CAP;
and Euc. I. 26 to show that
$\triangle$ APX $\equiv$ $\triangle$ BQY $\equiv$ $\triangle$ CRZ.
Hence show that
YZ = ZX = XY.

6. **Any point on the bisector of an angle is equally distant from the arms of the angle.**

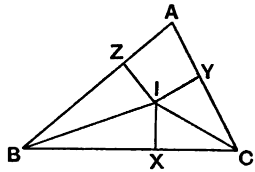


Let BI bisect $\angle$ ABC.
Draw IZ, IX $\perp$ AB, BC.
Use Euc. I. 26 to show IX = IZ.

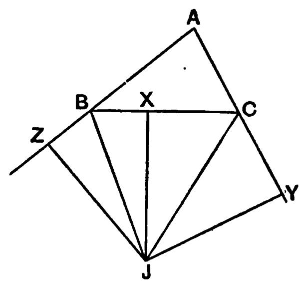
DEFINITION.—*The length of the perpendicular drawn from an external point to a straight line is called the* **Distance** *of the point from the line.*

7. The point in which the bisectors of two angles of a triangle meet is equally distant from the three sides of the triangle.

Let the bisectors of angles B and C meet in I.
Draw IX, IY, IZ ⊥ BC, CA, AB.
Use Euc. I. 26 to show that
IX = IZ, IX = IY.



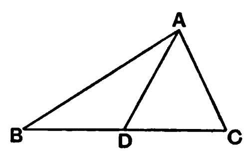
8. The point in which the bisectors of two exterior angles of a triangle meet is equally distant from the three sides of the triangle.



Use Euc. I. 26.

9. In the △ ABC, if D be the mid-point of BC and ∠ ADB be obtuse, then AB > AC.

Use Euc. I. 24 to compare
△s ADB, ADC.



10. The straight lines drawn through the extremities of the base of an isosceles triangle, equally inclined to it and terminated by the sides, are equal.

Use Euc. I. 26.

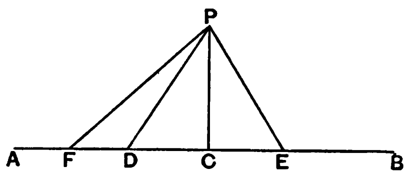
11. If two triangles have the same base, and if the straight line which joins their vertices bisect both vertical angles, show that both triangles are isosceles.

12. If, in the △ ABC, AB > AC and D be the mid-point of BC, then ∠ ADB shall be obtuse.

§ 8. (*Bookwork*, Euclid, I. 1-26.)

1. Of all straight lines which can be drawn to a given straight line from a given point outside, the perpendicular is the least; and, of the others, those which make equal angles with the perpendicular are equal, and that which makes a greater angle with the perpendicular is greater than that which makes a less angle.

(*A Standard Theorem.*)



Let P be the given point, and AB the given straight line, and let PC be $\perp$ AB, let PD, PE be drawn to AB making equal $\angle$s CPD, CPE, and let PF be drawn making $\angle$CPF > $\angle$CPD; it is required to prove

(1) PD > PC,
(2) PD = PE,
(3) PF > PD.

(1) In $\triangle$ PDC, PCD is a right angle;

$\therefore$ $\angle$PCD > $\angle$PDC; [Euc. I. 17.

$\therefore$ PD > PC. [Euc. I. 19.

(2) In $\triangle$s PDC, PEC $\left\{\begin{array}{l} PC = PC, \\ \angle CPD = \angle CPE, \quad \text{[Hypothesis.} \\ \angle PCD = \angle PCE; \quad \text{[Right angles.} \end{array}\right.$

$\therefore$ $\triangle$ PDC $\equiv$ $\triangle$ PEC; [Euc. I. 26.

$\therefore$ PD = PE.

(3) In $\triangle$ PDC, $\angle$PDF > rt. $\angle$PCD; [Euc. I. 16.

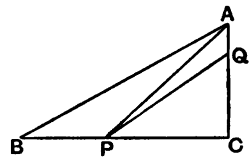
$\therefore$ $\angle$PDF is obtuse;

$\therefore$ $\angle$PFD is acute; [Euc. I. 17.

$\therefore$ PF > PD. [Euc. I. 19.

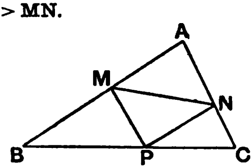
2. ABC is a triangle right-angled at C, and P, Q are points in BC, CA; show that AB > PQ.

Join AP.
Use Ex. 1 to show that
$AB > AP$,
and similarly
$AP > PQ$.



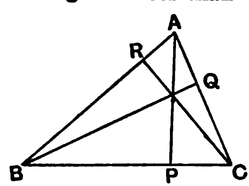
3. P is any point in the side BC of △ ABC; PM, PN are perpendiculars to AB, AC; show that BC > MN.

Use Euc. I. 19 and I. 20.



4. The three altitudes of any triangle are together less than the perimeter.

Use Ex. 1 to show that
$AB > AP$, etc.



5. ABC is a triangle, obtuse-angled at C, and D is any point in BC; show that AD > AC and < AB.

Use Euc. I. 17 and I. 19.

6. If the point A be equidistant from the point B and the straight line CD, any point in BA shall be nearer to B than to CD; but any point in BA produced beyond A shall be farther from B than from CD.

Use Euc. I. 20 and Ex. 1.

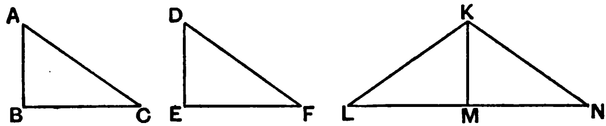
7. ABCD is a square, and F is any point on BC produced, show that AF > AC.

8. In the △ ABC, AP is the altitude from A; if BP > PC, show that AB > AC.

§ 9. (*Bookwork*, Euclid, I. 1-26.)

1. Two right-angled triangles shall be congruent if they have one of the sides containing the right angle in the one equal to one of the sides containing the right angle in the other, and also the hypotenuses equal.

(*A Standard Theorem.*)



Let ABC, DEF be two triangles, having AB=DE, AC=DF, and $\angle$s B, E rt. $\angle$s; it is required to prove that

$$\triangle ABC \equiv \triangle DEF.$$

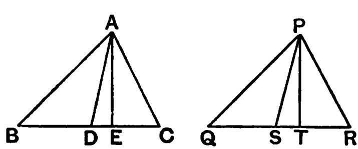
Let the $\triangle$s ABC, DEF be placed so that the equal sides AB, DE may coincide as KM, and the points C and F may fall on opposite sides of KM as L and N.

Then because $\angle$s B, E are right angles;
LMN will form one straight line; [Euc. I. 14.
and KL=KN, since AC=DF;
$\therefore \angle KLM = \angle KNM$; [Euc. I. 5.

$$\therefore \text{ in } \triangle\text{s } KLM, KNM \left\{ \begin{array}{ll} KM = KM, & \\ \angle KML = \angle KMN, & \text{[Right angles.} \\ \angle KLM = \angle KNM; & \text{[Proved above.} \end{array} \right.$$

$\therefore \triangle KLM \equiv \triangle KNM$, [Euc. I. 26.
or $\triangle ABC \equiv \triangle DEF$.

2. Two triangles shall be congruent if their bases, medians, and altitudes be respectively equal.



Use Ex. 1 to show $\triangle ADE \equiv \triangle PST$, and $\therefore$ DE=ST. Hence show that BE=QT, CE=RT and use Euc. I. 4 to show that $\triangle ABE \equiv \triangle PQT$, etc.

3. If two triangles have two sides and an altitude in each respectively equal, and both altitudes fall within or beyond the bases, the triangles shall be congruent.

Case I.

In either pair of figures, let

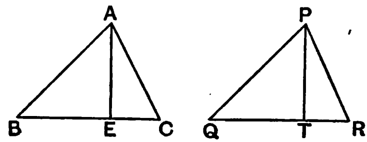
AB = PQ, AC = PR,

AE = PT.

Use Ex. 1 to show

△ABE ≡ △PQT,

△ACE ≡ △PRT.



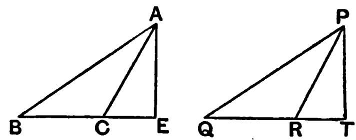
Case II.

Let AB = PQ, BC = QR,

AE = PT.

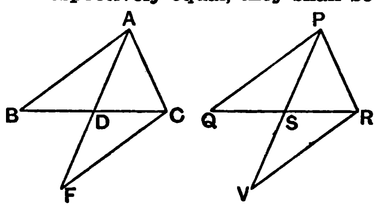
Use Ex. 1 to show

△ABE ≡ △PQT.



Then use Euc. I. 4.

4. If two triangles have two sides and a corresponding median in each respectively equal, they shall be congruent.



Case I.

Let AB = PQ, BC = QR (and ∴ BD = QS), AD = PS.

Use Euc. I. 8 to show △ ABD ≡ △ PQS,

and Euc. I. 4 to show △ ABC ≡ △ PQR.

Case II.

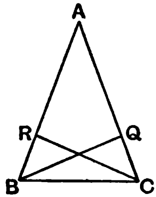
Let AB = PQ, AC = PR, AD = PS.

Produce AD, PS to F and V, so that DF = AD, SV = PS.

Join CF, RV. Prove CF = AB, etc.

Use Euc. I. 8 to show △ ACF ≡ △ PRV; Euc. I. 4 to show △ DAC ≡ △ SPR, and I. 8 to show △ ABC ≡ △ PQR.

5. If the altitudes drawn from the extremities of the base of a triangle be equal, the triangle shall be isosceles.



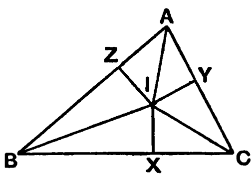
Use Ex. 1 to show

$\triangle$ BCR $\equiv \triangle$ CBQ,

and apply Euc. I. 6.

6. The bisectors of the three angles of a triangle are concurrent.

(*A Standard Theorem.*)



Let the bisectors of $\angle$s B and C meet in I.

Draw IX, IY, IZ $\perp$ BC, CA, AB.

Show as in § 7, Ex. 7,

IX=IY=IZ.

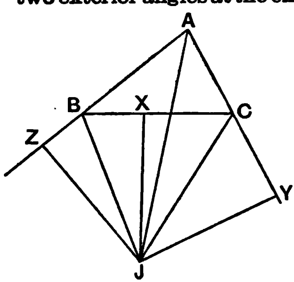
Use Ex. 1 to show

$\triangle$ AIZ $\equiv \triangle$ AIY,

and deduce that AI bisects $\angle$ A.

7. The bisectors of the vertical angle of a triangle and of the two exterior angles at the extremities of the base are concurrent.

(*A Standard Theorem.*)



Bisect exterior $\angle$s at B and C by BJ, CJ.

Draw JX, JY, JZ $\perp$ BC, CA, AB.

Use Euc. I. 26 to show

JX=JY=JZ.

Use Ex. 1 to show

$\triangle$ AJZ $\equiv \triangle$ AJY, etc.

8. A is the centre of a circle, and through B a point within the circle CD is drawn perpendicular to AB, meeting the circumference in C and D. Show that CB=DB.

Use Ex. 1.

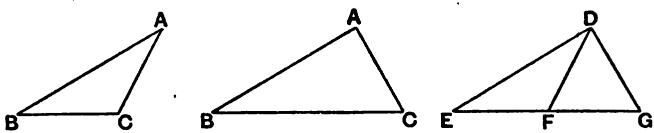
9. If the bisectors of the exterior angles of the $\triangle$ ABC be produced to form a $\triangle$ PQR ($\angle$ P opposite $\angle$ A, etc.), then AP, BQ, CR shall be concurrent, and shall also be perpendicular to QR, RP, PQ.

Use Exx. 7 and 6.

§ 10. (*Bookwork*, Euclid, I. 1-27.)

1. If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles equal which are opposite to one pair of equal sides, the angles opposite to the other pair of equal sides shall be either equal or supplementary; and, in the former case, the triangles shall be congruent.

(*A Standard Theorem.*)



Let ABC, DEF be two triangles, having AB=DE, AC=DF, and $\angle$ B= $\angle$ E; it is required to prove that

either $\angle$ C= $\angle$ F,

or $\angle$ C+ $\angle$ F=two right angles.

Let $\triangle$ ABC be applied to $\triangle$ DEF

so that A may fall on D, and AB along DE;

then B will fall on E, since AB=DE;

also BC will lie along EF, since $\angle$ B= $\angle$ E.

If the point C fall on F, then

$\triangle$ ABC $\equiv$ $\triangle$ DEF, [Euc. I. 4.

and $\angle$ C= $\angle$ F.

But if C do not fall on F, let it fall otherwise as at G.

Then DG (or AC)=DF;

$\therefore$ $\angle$ DFG= $\angle$ DGF.

But $\angle$ DFG+ $\angle$ DFE=two right angles;

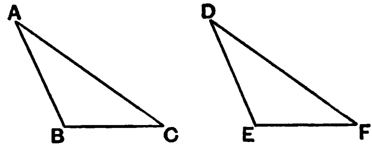
$\therefore$ $\angle$ DGF (or ACB)+ $\angle$ DFE=two right angles.

i.e. $\angle$ C+ $\angle$ F=two right angles.

Note.—(1) If the equal angles B and E are right angles the case becomes that of § 9, Ex. 1.

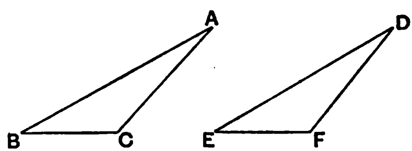
(2) If the conditions are known to be such that angles C, F cannot be supplementary, the triangles must be congruent.

2. Two $\triangle$s ABC, DEF have AB=DE, AC=DF, and the obtuse $\angle$ B=$\angle$ E; show that $\triangle$ ABC$\equiv$$\triangle$ DEF.



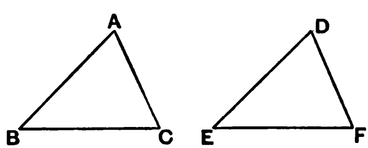
Use Ex. 1 to show that $\angle$s C, F must be either equal or supplementary, and use Euc. I. 17 to show that both are acute, and must therefore be equal.

3. Two $\triangle$s ABC, DEF have AB=DE, AC=DF, and $\angle$ B=$\angle$ E; also the $\angle$s C, F are known to be both obtuse; show that $\triangle$ ABC$\equiv$$\triangle$ DEF.



Use Ex. 1, and show that $\angle$s C and F cannot be supplementary.

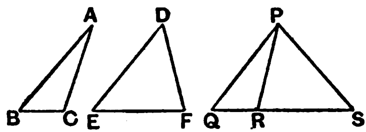
4. Two $\triangle$s ABC, DEF, have the sides AB, AC respectively equal to DE, DF, $\angle$ C=$\angle$ F, and it is known that AC is not > AB; show that $\triangle$ ABC$\equiv$$\triangle$ DEF.



Use Ex. 1, and show that neither the $\angle$ B nor the $\angle$ E can be obtuse, as in that case we should have, by Euc. I. 19,

$$AC>AB \text{ or } DF>DE.$$

5. Two $\triangle$s ABC, DEF have AB=DE, AC=DF, and $\angle$ C+$\angle$ F equal to two right angles; show that $\angle$ B=$\angle$ E.



Place the triangles so that AC and DF coincide as PR, while the points B and E lie on opposite sides of PR as Q and S; then

QRS is a straight line, [Euc. I. 14.

and $\angle$ Q=$\angle$ S; [Euc. I. 5.

that is, $\angle$ B=$\angle$ E.

6. Two $\triangle$s ABC, DEF have AB=DE, AC=DF, and $\angle$ B=$\angle$ E; also the $\angle$s C, F are known to be both acute; show that $\triangle$ ABC$\equiv$$\triangle$ DEF.

Use Ex. 1.

7. Two $\triangle$s ABC, DEF have AB=DE, $\angle$ B=$\angle$ E and $\angle$ C supplementary to $\angle$ F; show that AC=DF.

Use superposition and Euc. I. 6.

8. ABCD is a quadrilateral having AB=CD, and the obtuse $\angle$ A=$\angle$ C; show that ABCD is a parallelogram.

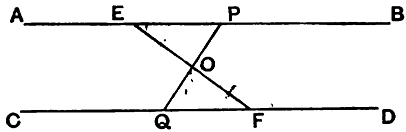
Use Ex. 1 and Euc. I. 27.

9. ABCD is a quadrilateral having AB=CD, and the acute $\angle$ A=$\angle$ C; show that ABCD is or is not a parallelogram according as the $\angle$s ADB, DBC are equal or supplementary.

Use Ex. 1 to compare the $\triangle$s ABD, CDB.

§ 11. (*Bookwork*, EUCLID, I. 1-29.)

1. If a straight line intersect two parallel straight lines, any line drawn through its mid-point, and terminated by the parallel lines, shall be bisected at that point.



Let AB, CD be two parallel straight lines intersected by EF, of which O is the mid-point, and let POQ meet AB, CD in P, Q; it is required to prove that

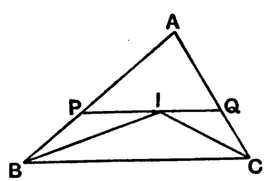
PO = QO.

In $\triangle$s PEO, QFO $\left\{\begin{array}{l} \text{EO} = \text{FO}, \\ \angle\text{PEO} = \angle\text{QFO}, \\ \angle\text{EPO} = \angle\text{FQO}; \end{array}\right.$ [Hypothesis. / [Euc. I. 29. / [Euc. I. 29.

$\therefore$ $\triangle$ PEO $\equiv$ $\triangle$ QFO [Euc. I. 26.

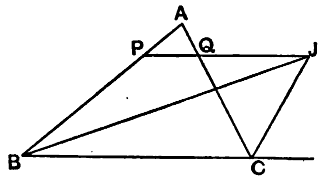
$\therefore$ PO = QO.

2. ABC is a triangle, and through I, the point of intersection of the bisectors of $\angle$s B and C, PIQ is drawn $\parallel$ BC, meeting AB, AC in P, Q; show that PQ=PB+QC.



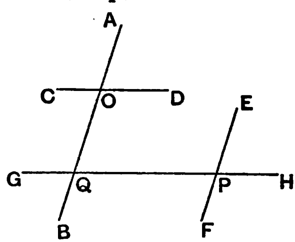
Use Euc. I. 29 and I. 6.

3. ABC is a triangle, having AB > AC, and through J, the point of intersection of the bisectors of ∠B and exterior angle at C, PQJ is drawn parallel to BC, meeting AB, AC in P, Q; show that PQ = PB − QC.



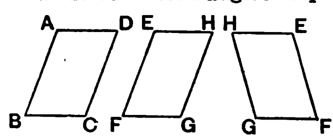
Use Euc. I. 29 and I. 6.

4. If two intersecting straight lines be respectively parallel to two other intersecting straight lines, any angle made by the first pair shall be either equal or supplementary to any angle made by the second pair.



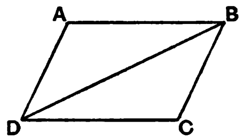
O and P are the points of intersection.
Let AB, GH meet in Q.
Use Euc. I. 29 to show that
∠AOD = ∠OQP = ∠EPH = ∠GPF. Apply Euc. I. 13.

5. Two parallelograms shall be congruent if they have two adjacent sides and the contained angles respectively equal.



Let AD = EH, AB = EF, ∠A = ∠E.
Use Euc. I. 29 to show ∠D = ∠H, ∠B = ∠F,
and prove by superposition.

6. If the opposite sides of a quadrilateral be equal it shall be a parallelogram.



Join BD. Use Euc. I. 8 to show

$\triangle ABD \equiv \triangle CDB$,

and apply Euc. I. 27 to show

$AB \parallel DC$ and $AD \parallel BC$.

Cor.—Every rhombus is a parallelogram.

7. If from any point in the bisector of an angle straight lines be drawn parallel to the arms of the angle and produced to meet the arms, the figure thus formed shall be a rhombus.

Use Euc. I. 29 and I. 6.

8. If one angle of a parallelogram be bisected by a diagonal, then all the angles shall be bisected by diagonals.

Use Euc. I. 29, etc.

9. ABCD is a quadrilateral having $BC \parallel AD$. If AC bisects $\angle A$ and BD bisects $\angle D$, show that $AB = BC = CD$.

10. ABC is a triangle having $AC > AB$; through J, the point of intersection of the bisectors of $\angle B$ and exterior $\angle C$, QPJ is drawn $\parallel$ BC, meeting BA, CA, produced in P, Q; show that $PQ = QC - PB$.

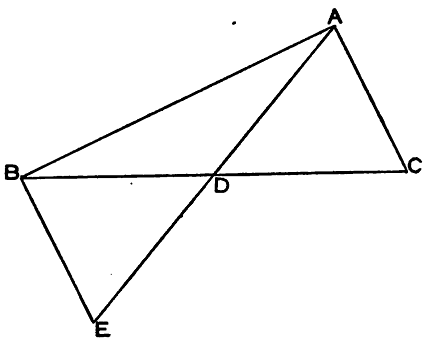
§ 12. (*Bookwork*, Euclid, I. 1-32.)

1. In a right-angled triangle the mid-point of the hypotenuse is equally distant from the three angular points.

(*A Standard Theorem.*)

Let ABC be a triangle right-angled at A, and let D be the mid-point of BC; it is required to prove that

AD = BD = CD.



Produce AD to E, so that DE = AD. Join BE.

In △s ADC, EDB $\left\{\begin{array}{l} AD = ED, \\ DC = DB, \\ \angle ADC = \angle EDB; \end{array}\right.$ [Construction. [Hypothesis. [Euc. I. 15.

∴ △ ADC ≡ △ EDB; [Euc. I. 4.

∴ AC = BE,

and ∠ ACD = ∠ EBD.

But ∠ ACB + ∠ ABC = a right angle; [Euc. I. 32.

∴ ∠ ABE = a right angle.

In △s BAC, ABE $\left\{\begin{array}{l} BA = AB, \\ AC = BE, \\ \angle BAC = \angle ABE; \end{array}\right.$ [Proved above. [Right angles.

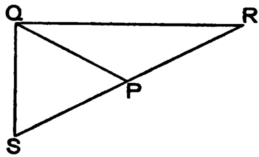
∴ △ BAC ≡ △ ABE; [Euc. I. 4.

∴ BC = AE;

∴ their halves are equal, or

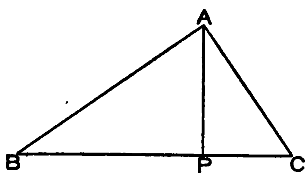
AD = BD = CD.

2. PQR is an isosceles triangle having PQ=PR, and RP is produced its own length to S; show that RQS is a right angle.



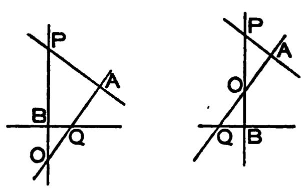
Use Euc. I. 5 to show that
$\angle Q = \angle R + \angle S$
and apply Euc. I. 32.

3. ABC is a triangle right-angled at A, and AP is perpendicular to BC; show that triangles ABC, PAC, PBA are equiangular to each other.



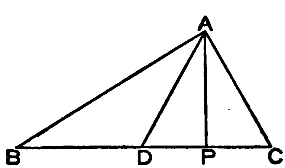
Show that $\triangle$s ABC, PAC have one angle common, and have each a right angle; and $\therefore$ by Euc. I. 32 their third angles are equal, etc.

4. If two intersecting straight lines be respectively perpendicular to two other intersecting straight lines; any angle made by the first pair shall be either equal or supplementary to any angle made by the second pair.



Let $PA \perp QA$, $PB \perp QB$.
Show that in
$\triangle$s POA, QOB,
two angles are equal, and
$\therefore \angle APO = \angle BQO$ [Euc. I. 32.

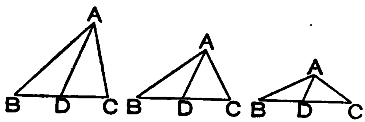
5. ABC is a triangle right-angled at A, AD is the median, and AP the altitude from A; show that $\angle$ DAP is equal to the difference of the $\angle$s B and C.



Use § 12, Ex. 1 to show that
$\angle DAC = \angle C$,
and show as in Ex. 3 that
$\angle PAC = \angle B$.

6. Any angle of a triangle is acute, right, or obtuse according as the median from it is greater than, equal to, or less than half the opposite side.

Use Euc. I. 18 or 5, and 32.



7. AOC, BOD are two intersecting straight lines; AB, CD are joined, and the $\angle$s B, C are bisected by BE, CE; show that $\angle E = \frac{1}{2}(\angle A + \angle D)$.

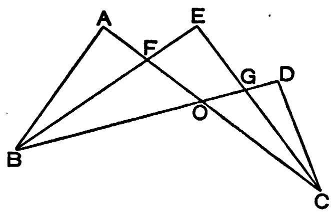
In $\triangle$s ABF, CEF

$\angle A + \frac{1}{2}\angle B = \angle E + \frac{1}{2}\angle C$.

In $\triangle$s CDG, BEG,

$\angle D + \frac{1}{2}\angle C = \angle E + \frac{1}{2}\angle B$.

Add results, etc.



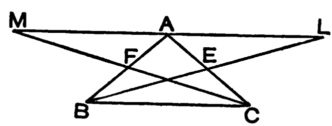
8. F and E are the mid-points of AB, AC; BE and CF are produced to L and M so that EL=BE, FM=CF; show that M, A, L are collinear.

Use Euc. I. 4 to show that

$\angle MAF = \angle FBC$,

and $\angle LAE = \angle ECB$,

and apply Euc. I. 32 and I. 14.

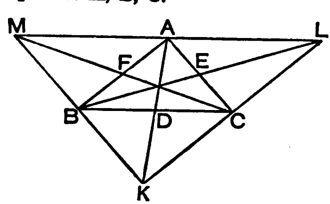


9. Each of the medians AD, BE, CF is produced its own length to K, L, M; show that the sides of the $\triangle$KLM pass through and are bisected by the points A, B, C.

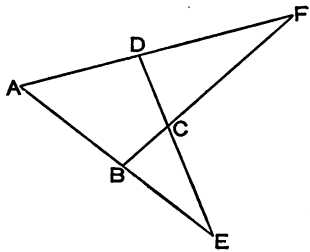
Show as in previous Ex. that

M, A, L are collinear

and MA=BC=AL, etc.

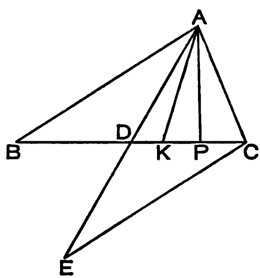


10. ABCD is a quadrilateral, having ∠ABC = ∠ADC; AB and DC meet in E; AD and BC in F; show that ∠AED = ∠AFB.



Use Euc. I. 32 to compare angles of △s AED, AFB.

11. If two sides of a triangle be unequal, and if from their point of intersection three straight lines be drawn, namely the bisector, the median, and the altitude, the first shall be intermediate, both in magnitude and position, to the other two.



In △ ABC, AB > AC.

Let AD, AK, AP be the median, bisector, and altitude.

Produce AD to E so that DE = AD.

Use Euc. I. 4 to show △ ADB ≡ △ EDC, and EC = AB

∴ EC > AC.

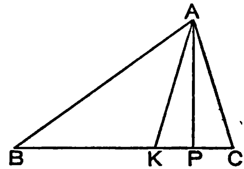
Use Euc. I. 18 to show ∠ DAC > ∠ DAB;

∴ AD falls within ∠ BAK.

Use Euc. I. 18 and I. 32 to show ∠ BAP > ∠ CAP;

∴ AP falls within ∠ CAK. Apply § 8, Ex. 1.

12. **In any triangle the angle contained by the bisector of an angle and the altitude drawn from that angle is equal to half the difference of the other angles.**



Use Euc. I. 32.

$\angle AKP = \angle B + \frac{1}{2}\angle A$;

$\angle APK = \frac{1}{2}(\angle A + \angle B + \angle C)$, etc.

13. **In the $\triangle ABC$, D is the mid-point of BC, and BL, CM are the altitudes from B and C; show that $\triangle DLM$ is isosceles.**

Use Ex. 1.

14. **ABC is any triangle, and from A and C, AD, CE are drawn equal and perpendicular to AB, BC respectively. From D and E, DF, EG are drawn perpendicular to AC, or AC produced. Show that DF+EG=AC when angles A and C are both acute, but that DF ~ EG=AC when one of them is obtuse.**

Draw $BH \perp AC$, and use Euc. I. 32 and I. 26.

15. **ABC is a triangle in which the $\angle B$ is double the $\angle C$, and the altitude AP falls within BC; if AB be produced to Q so that BQ=BP, then QP produced shall bisect AC.**

Use Euc. I. 32 and I. 6.

16. **If through three points in the sides of a triangle three straight lines be drawn making equal angles, towards the same parts (that is, in the same direction of circular rotation), with the three sides of a triangle, they shall form a triangle equiangular with the given triangle.**

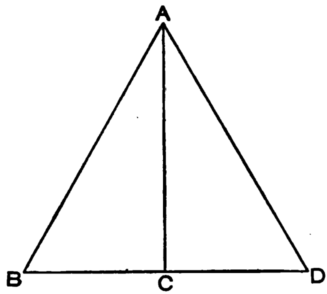
Use Euc. I. 15 and I. 32.

17. **If each of the angles at the base of an isosceles triangle be one-fourth the vertical angle, any line drawn perpendicular to the base shall form an equilateral triangle with the sides (produced where necessary).**

18. **If in the sides of a given square four points be taken at equal distances from the angular points toward the same parts, the figure contained by the straight lines which join these points shall be a square.**

§ 13. (*Bookwork*, EUCLID, I. 1-32.)

1. If in a right-angled triangle one of the acute angles be twice the other, then the hypotenuse shall be equal to twice the side opposite the smaller acute angle.



Let ABC be a triangle, right-angled at C and having $\angle ABC = 2 \angle BAC$; it is required to prove that

$$AB = 2BC.$$

Produce BC to D so that CD = BC. Join AD.

In $\triangle$s ACB, ACD $\left\{\begin{array}{l} AC = AC, \\ CB = CD, \quad \text{[Construction.} \\ \angle ACB = \angle ACD; \quad \text{[Right angles.} \end{array}\right.$

$\therefore \triangle ACB \equiv \triangle ACD$; [Euc. I. 4.

$\therefore \angle BAC = \angle DAC, \quad \angle ABC = \angle ADC$;

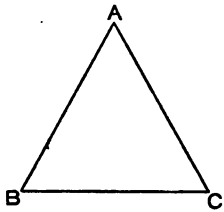
$\therefore \angle BAD = 2 \angle BAC = \angle ABC = \angle ADC$;

$\therefore BD = DA = AB$; [Euc. I. 6, Cor.

or $AB = 2BC$.

NOTE.—It is usual to measure angles by degrees, a right angle being 90°. The sum of the three angles of any triangle is therefore 180°, and each angle of an equilateral triangle is 60°.

2. If in an isosceles triangle any one angle be 60°, the triangle shall be equilateral.



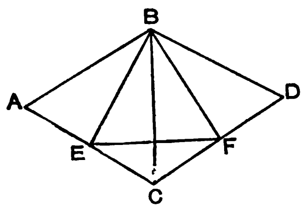
(1) Let $AB = AC$, and $\angle B = 60°$.

(2) Let $AB = AC$ and $\angle A = 60°$.

Use Euc. I. 5 and Euc. I. 32.

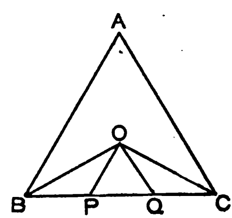
3. ABC, DBC are two equilateral triangles on opposite sides of the same base; BE, BF are the bisectors of the angles ABC, DBC, meeting AC, CD in E, F; show that △ BEF is equilateral.

Use Euc. I. 4 to show
$\triangle ABE \equiv \triangle CBE$;
And Euc. I. 4 or I. 26 to show
$\triangle BCE \equiv \triangle BCF$;
$\therefore \angle EBF = \angle ABC$ and $EB = BF$,
(the case discussed in Ex. 2).

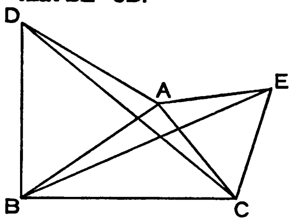


4. ABC is an equilateral triangle; BO, CO bisect the angles B, C; OP, OQ are parallel to AB, AC, meeting BC in P, Q; show that BC is trisected in P and Q.

Use Euc. I. 29 and I. 32 to show
OPQ an equilateral $\triangle$.
Use Euc. I. 29 and I. 6 to show
$BP = OP$, etc.



5. ABC is any triangle, and on AB, AC equilateral △s ADB, AEC are described both externally (or both internally); show that BE = CD.

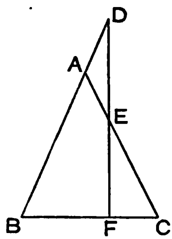


Use Euc. I. 4 to show

△ BAE ≡ △ DAC.

A similar method applies to the case where the equilateral triangles are described internally.

6. ABC is an isosceles triangle, and DEF is perpendicular to BC, meeting BA produced in D, AC in E, and BC in F; show that △ ADE is isosceles.



Show by Euc. I. 32 that

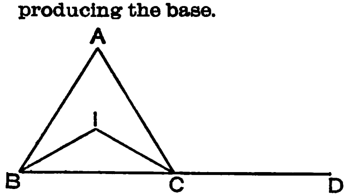
∠ ADE is the complement of ∠ B,

and by Euc. I. 15 and 32 that

∠ AED is the complement of ∠ C, etc.

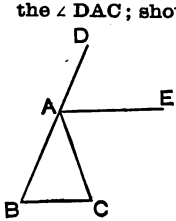
Examine the case in which E is on AC produced.

7. The angle contained by the bisectors of the equal angles of an isosceles triangle is equal to the exterior angle formed by producing the base.



Show that the three angles of △ IBC = the three angles at point C.

8. In the △ ABC, BA is produced to D, and AE bisects the ∠ DAC; show that



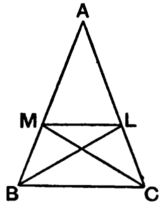
(1) If AB = AC, AE ∥ BC;

(2) If AE ∥ BC, AB = AC.

(1) Use Euc. I. 5, 32, and 27.

(2) Use Euc. I. 29 and 6.

9. ABC is an isosceles triangle, and the bisectors of the equal angles B, C meet the opposite sides in L, M; show that BM=ML=CL.



Use Euc. I. 26 to show AM=AL, and hence by Euc. I. 32 and I. 27, prove ML ∥ BC. Then use Euc. I. 29 to show ∠ MLB=∠ LBC=∠ LBM, and apply Euc. I. 6.

10. If from AB, AC, the equal sides of an isosceles triangle, equal lengths, AD, AE, be cut off, show that DE ∥ BC.

Use Euc. I. 32 and I. 28.

11. The bisectors of the exterior angles at the base of an isosceles triangle contain an angle equal to one of the interior angles at the base. M

12. ABC is an isosceles right-angled triangle. From AC, the hypotenuse, AD is cut off=AB, and DE is drawn ⊥ AC, meeting BC in E; show that BE=ED=CD.

Join BD, and use Euc. I. 5, 32, and 6.

13. In the △ ABC, AK, the bisector of ∠ A, meets BC in K: show that ∠ AKC=one-half the sum of ∠ B and the exterior angle at C.

14. ABC is an isosceles triangle having AB=AC; from BA, BC, equal parts, BD, BE, are cut off, and DE produced meets AC in F; show that three times the ∠ ADE=four right angles + ∠ AFE.

15. If the bisectors of the angles of the △ ABC meet in I, show that the angles BIC, CIA, AIB are obtuse.

16. If the bisectors of the angles of the △ ABC meet in I, and AI be produced to K, and IX be drawn from I perpendicular to BC, then ∠ BIK=∠ CIX.

17. O is a point within △ ABC at which the three sides subtend equal angles. From A perpendiculars are drawn to OB and OC, and the line is drawn joining the feet of these perpendiculars. Similar lines are drawn corresponding to B and C. Show that these three lines form an equilateral triangle.

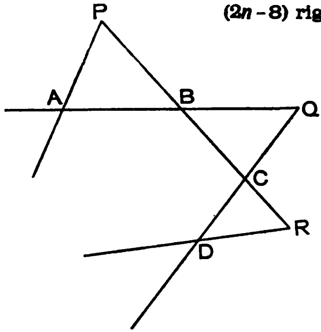
Show that OA bisects the angle between OB produced and OC produced, and hence prove the three lines respectively perpendicular to OA, OB, OC.

§ 14. (*Bookwork*, Euclid, I. 1-32, Corollaries.)

Definition.—*A* **convex** *polygon is one in which each of the interior angles is less than two right angles.*

Thus ABIC in § 13, Ex. 7, is *not* a convex polygon.

1. If the alternate sides of a convex polygon of n sides be produced to meet, the sum of the angles thus formed shall be $(2n-8)$ right angles.



Let ABCD . . be a convex polygon of n sides, and let P, Q, R, . . . be the angles formed by producing the alternate sides to meet; it is required to prove that

$$\angle P + \angle Q + \angle R + \ldots = (2n-8) \text{ rt. } \angle \text{s}.$$

All the angles of the n $\triangle$s PAB, QBC, RCD, . . . $= 2n$ rt. $\angle$ s.

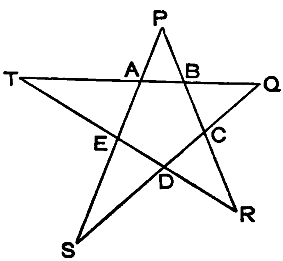
But $\angle PAB + \angle QBC + \angle RCD + \ldots = 4$ rt. $\angle$ s.

[Euc. I. 32, Cor. ii.

Similarly, $\angle PBA + \angle QCB + \angle RDC + \ldots = 4$ rt. $\angle$ s.;

$\therefore$ by subtraction, $\angle P + \angle Q + \angle R + \ldots = (2\,n-8)$ rt. $\angle$ s.

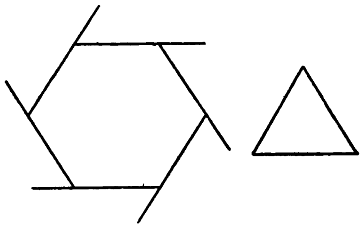
2. The alternate sides of a pentagon are produced to meet; show that the sum of the angles thus formed is two right angles.



Here $2n-8=10-8=2$.

Similar results may be obtained for other polygons.

3. Show that the exterior angle of an equiangular hexagon equals the interior angle of an equilateral triangle.



Use Euc. I. 32, Cor. ii.; and Euc. I. 32.

4. The opposite sides of an equiangular hexagon are parallel.

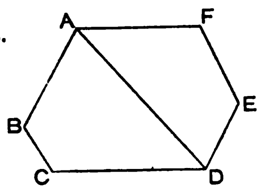
Join AD.

The $\angle$s of ABCD $=$ $\angle$s of AFED $=$ 4 rt. $\angle$s.
[Euc. I. 32, Cor. i.

$\therefore \angle FAD + \angle ADE = \angle DAB + \angle ADC.$

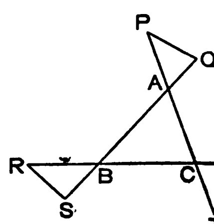
But $\angle FAD + \angle DAB = \angle ADE + \angle ADC.$

Hence prove $\angle FAD = \angle ADC$, and apply Euc. I. 27.

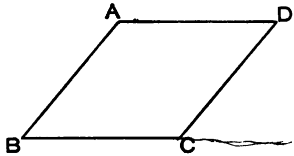


5. Points are taken on the produced sides of a triangle and joined as in the figure; show that the exterior angles of the crossed polygon thus formed equal eight right angles.

The exterior angle at P is that contained by QP produced and PT.
Show interior angles of polygon
$=$ the angles of
$\triangle$s APQ, BRS, CTU
$-$ angles of $\triangle$ ABC
$=$ four right angles,
and apply Euc. I. 13.

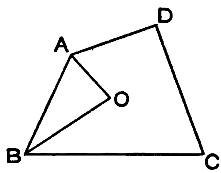


6. A quadrilateral whose opposite angles are equal is a parallelogram.



Use Euc. I. 32, Cor. i., and I. 28.

7. The angle contained by the bisectors of two adjacent angles of a quadrilateral equals half the sum of the remaining angles.



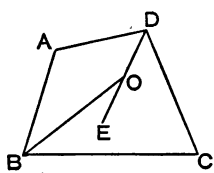
Use Euc. I. 32, Cor. i. to show

$\frac{1}{2}\angle A+\frac{1}{2}\angle B+\frac{1}{2}\angle C+\frac{1}{2}\angle D=$two right angles.

And Euc. I. 32 to show

$\frac{1}{2}\angle A+\frac{1}{2}\angle B+\angle O=$two right angles, etc.

8. The angle contained by the bisectors of two opposite angles of a quadrilateral equals half the difference of the remaining angles.



As in Ex. 7,

$\frac{1}{2}\angle A+\frac{1}{2}\angle B+\frac{1}{2}\angle C+\frac{1}{2}\angle D=$two right angles;

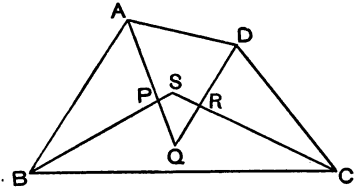
but $\angle A+\frac{1}{2}\angle B+\angle BOD+\frac{1}{2}\angle D=$four right angles;

$\therefore$ by subtraction $\angle BOD+\frac{1}{2}\angle A-\frac{1}{2}\angle C=$two right angles.

Hence show $\angle BOE=\frac{1}{2}(\angle A-\angle C)$.

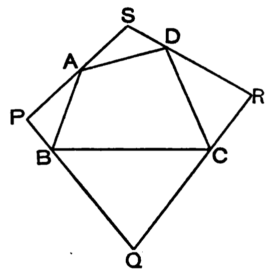
9. The bisectors of the angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.

Use the method of Ex. 7, Euc. I. 32, Cor. i., and Euc. I. 15.



10. The bisectors of the exterior angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary.

The proof resembles that of Ex. 9.



11. If the angles of a quadrilateral taken in order be proportional to the numbers 1, 2, 3, 4, show that two opposite sides must be parallel.

Use Euc. I. 32, Cor. i., and Euc. I. 28.

12. Show that three regular hexagons can be placed so as to have a common angular point and completely fill up the space round that point.

Use Euc. I. 32, Cor. i.

13. Show that a square, a regular hexagon, and a regular dodecagon can be placed so as to have a common angular point and completely fill up the space round that point.

14. If a rectilinear figure have fifteen equal angles, each of them shall measure 156°.

15. A seven-rayed star is formed by joining the alternate angles of a heptagon; show that the sum of the angles of the star is six right angles.

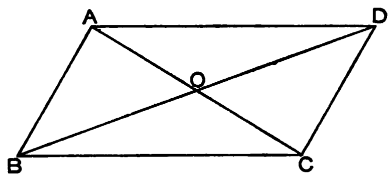
§ 15. (*Bookwork*, Euclid, I. 1-33.)

1. The diagonals of a parallelogram bisect each other; and conversely, if the diagonals of a quadrilateral bisect each other, the quadrilateral shall be a parallelogram.

(*A Standard Theorem.*)

(1) Let ABCD be a parallelogram, and let AC, BD intersect in O; it is required to prove that

$$AO = CO, \ BO = DO.$$



In $\triangle$s AOB, COD $\left\{\begin{array}{l} AB = CD, \\ \angle BAO = \angle DCO, \\ \angle ABO = \angle CDO; \end{array}\right.$ [Euc. I. 34. / [Euc. I. 29. / [Euc. I. 29.

$\therefore \triangle AOB \equiv \triangle COD,$ [Euc. I. 26.

$\therefore AO = CO, \ BO = DO.$

(2) Let the diagonals AC, BD of the quadrilateral ABCD meet in O, and let AO = CO, BO = DO; it is required to prove that

$$BA \parallel CD, \ AD \parallel BC.$$

In $\triangle$s AOB, COD $\left\{\begin{array}{l} AO = CO, \\ OB = OD, \\ \angle AOB = \angle COD; \end{array}\right.$ [Hypothesis. / [Hypothesis. / [Euc. I. 15.

$\therefore \triangle AOB \equiv \triangle COD,$ [Euc. I. 4.

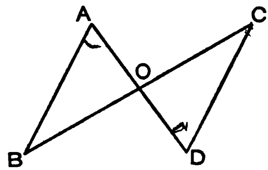
$\therefore \angle OAB = \angle OCD;$

$\therefore BA \parallel CD,$ [Euc. I. 27.

Similarly, $AD \parallel BC$.

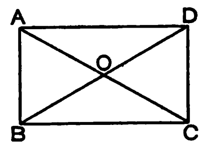
2. If two straight lines be equal and parallel, the straight lines which join their extremities towards opposite parts shall bisect each other.

Use Euc. I. 29 and I. 26.



3. If two equal straight lines bisect each other, the figure formed by joining their extremities shall be a rectangle.

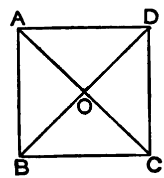
Use Euc. I. 5 to show
$\angle BAD = \angle ABD + \angle ADB$,
and apply Euc. I. 32.



4. If two equal straight lines bisect each other at right angles, the figure formed by joining their extremities shall be a square.

Use Euc. I. 4, I. 5, and I. 32.

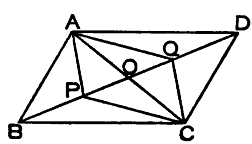
Cor.—A rectangle, whose diagonals contain a right angle, is a square.



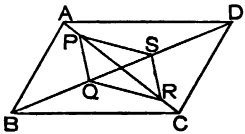
5. ABCD is a parallelogram, and P and Q are taken on BD, so that BP=DQ; show that APCQ is a parallelogram.

Use Ex. 1 to show that AC, PQ bisect each other in O, etc.

Examine the case in which P and Q are on BD produced.

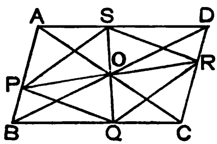


6. ABCD is a parallelogram; P, R are points on AC, such that AP=CR; Q, S are points on BD, such that BQ=DS; show that PQRS is a parallelogram.



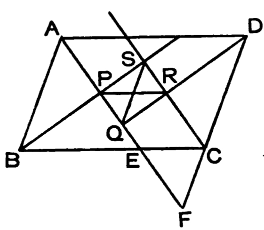
Show that PR and QS bisect each other, etc.

7. P, Q, R, S are points on the sides AB, BC, CD, DA of a parallelogram, such that AP=CR, BQ=DS; show that PQRS is a parallelogram.



Use Euc. I. 33 to show APCR a ▱;
∴ AC and PR bisect each other in O.
Similarly,
BD and QS bisect each other in O;
∴ PR and QS bisect each other, etc.

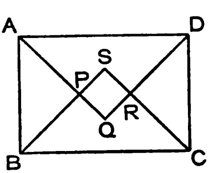
8. The bisectors of the angles of a parallelogram form a rectangle whose diagonals are parallel to the sides of the parallelogram.



Show as in § 14, Ex. 7 that
∠s P, Q, R, S right ∠s.
Show △APB≡△EPB,
and △APB≡△CRD;
hence show PE= and ∥ RC,
and by Euc. I. 33, PR ∥ EC.
Similarly, show
SC= and ∥ QF, etc.

Examine the case in which the exterior angles are bisected.

9. The bisectors of the angles of a rectangle form a square.

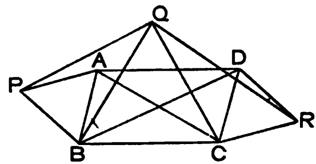


Show, as in Ex. 8, that PQRS is a rectangle, and that its diagonals are parallel to the sides of ABCD, and use Ex. 4, Cor.

Examine the case of exterior angles.

10. On the sides AB, BC, CD of a ▱ ABCD equilateral triangles are described, that on BC inwards, and those on AB, CD outwards; show that the distances of the vertices of the latter triangles from that of the former triangle are equal to the diagonals of the parallelogram.

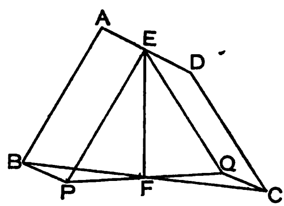
Show $\angle PBQ = \angle ABC$.
Use Euc. I. 4 to show that
$\triangle PBQ \equiv \triangle ABC$,
and $\triangle BCD \equiv \triangle QCR$.



11. ABCD is a quadrilateral, in which AB is not parallel to DC, and E, F are the mid-points of AD, BC; show that

$$AB + CD > 2EF.$$

Complete ▱s BAEP, EDCQ,
and show BP = and ∥ CQ.
Apply Ex. 1 to prove PF = FQ,
and use § 5, Ex. 1.



12. If two opposite sides of a quadrilateral be equal, they shall be equally inclined to the straight line joining the mid-points of the other two sides.

Use construction of previous Ex. and § 3, Ex. 1.

13. If a parallelogram be inscribed in another parallelogram, the point of intersection of its diagonals shall coincide with that of the first parallelogram.

Use Euc. I. 26 and the method of Ex. 7.

14. If the opposite sides of a hexagon be equal and parallel, the three diagonals which join the opposite angles shall be concurrent.

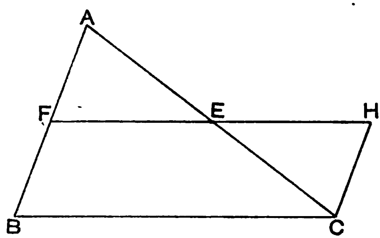
Use Ex. 1.

15. In the △s ABC, DEF, AB ∥ and = DE, BC ∥ and = EF; show that CA ∥ and = FD.

§ 16. (*Bookwork*, EUCLID, I. 1-34.)

1. The straight line which joins the mid-points of two sides of a triangle is parallel to the base and equal to half of it; and, conversely, the straight line drawn through the mid-point of a side of a triangle parallel to the base bisects the other side.

(A Standard Theorem.)



Let F, E be the mid-points of AB, AC; it is required to prove that

$$FE \parallel BC \text{ and} = \tfrac{1}{2}BC.$$

Through C draw CH $\parallel$ BA meeting FE produced in H.

In $\triangle$s AEF, CEH $\begin{cases} AE = CE, & \text{[Hypothesis.} \\ \angle AEF = \angle CEH, & \text{[Euc. I. 15.} \\ \angle AFE = \angle CHE; & \text{[Euc. I. 29.} \end{cases}$

$\therefore \triangle AEF \equiv \triangle CEH$; [Euc. I. 26.

$\therefore FE = EH = \tfrac{1}{2}FH$,

and $CH = AF$.

But $AF = BF$; [Hypothesis.

$\therefore CH = BF$.

And since CH $\parallel$ and $=$ BF,

$\therefore FH \parallel$ and $= BC$; [Euc. I. 33.

$\therefore FE \parallel BC$ and $= \tfrac{1}{2}BC$.

Conversely, let F be the mid-point of AB and FE || BC, it is required to prove that

$$AE = CE$$

Through C draw CH || BA to meet FE produced in H.

Then FC is, by construction, a parallelogram;

$$\therefore CH = BF = AF.$$

In $\triangle$s AFE, CHE $\begin{cases} AF = CH, & \text{[Proved above.} \\ \angle AEF = \angle CEH, & \text{[Euc. I. 15.} \\ \angle AFE = \angle CHE; & \text{[Euc. I. 29.} \end{cases}$

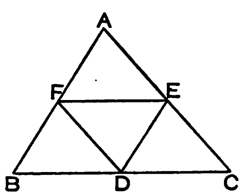
$\therefore \triangle AEF \equiv \triangle CEH,$ [Euc. I. 26.

$\therefore AE = CE.$

These theorems may also be proved by means of Euc. I. 38 and I. 39.

2. The straight lines which join the mid-points of the sides of a triangle divide it into four congruent triangles.

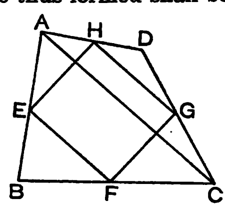
(*A Standard Theorem.*)



Use Ex. 1 and Euc. I. 8.

3. If the mid-points of the adjacent sides of any quadrilateral be joined, the figure thus formed shall be a parallelogram.

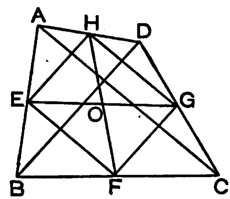
(*A Standard Theorem.*)



Use Ex. 1 to show that

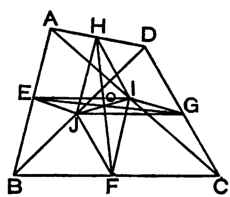
EF and HG are each || AC and $= \frac{1}{2}$AC, and apply Euc. I. 33.

4. The straight lines joining the mid-points of the opposite sides of any quadrilateral bisect each other.



Use Ex. 3 to show that EFGH is a parallelogram, etc.

5. The straight line which joins the mid-points of the diagonals of a quadrilateral and the straight lines which join the mid-points of its opposite sides are concurrent.



Use Ex. 3 to show that EJGI is a parallelogram, and hence that

EG and IJ bisect each other.

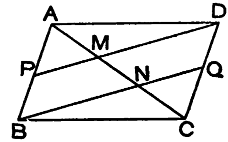
Similarly show that HF and IJ bisect each other, and deduce that

EG, IJ, FH are concurrent.

Observe that Ex. 3 is here seen to apply to the *crossed* quadrilateral ABDC.

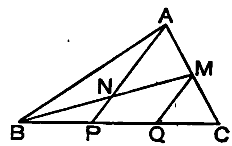
6. P and Q are the mid-points of the sides AB and CD of the ▱ ABCD; show that PD and BQ trisect AC.

Use Euc. I. 33 to show that PD ∥ BQ.
Use Ex. 1 to show that PM bisects AN, and that QN bisects MC.



7. ABC is a triangle; AC is bisected in M and BM is bisected in N; AN meets BC in P, and MQ is drawn ∥ AP to meet BC in Q; show that BP=PQ=QC.

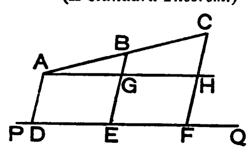
Use Ex. 1 to show that MQ bisects PC, and NP bisects BQ.



8. A, C are points on the same side of PQ, and B is the mid-point of AC; through A, B, C parallel straight lines are drawn, meeting PQ in D, E, F; show that AD+CF=2 BE.

(*A Standard Theorem.*)

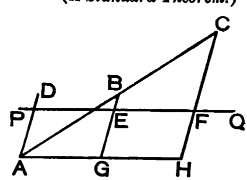
Draw AGH ∥ PQ.
Use Ex. 1 and Euc. I. 34 to show
CF−HF=2(BE−GE),
or CF−AD=2(BE−AD); etc.



9. A, C are points on opposite sides of PQ, and B is the mid-point of AC; through A, B, C parallel straight lines are drawn, meeting PQ in D, E, F; show that AD∼CF=2 BE.

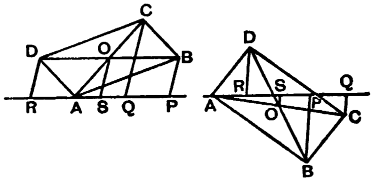
(*A Standard Theorem.*)

Draw AGH ∥ PQ.
Use Ex. 1 and Euc. I. 34 to show
CF+FH=2(BE+EG); etc.



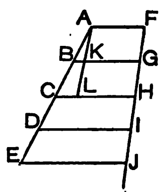
Observe that this may be considered as a particular case of the previous Ex.

10. ABCD is a parallelogram; BP, CQ, DR are parallel straight lines meeting a straight line through A in P, Q, R; show that CQ=BP±DR according as the line does not or does cut the parallelogram.



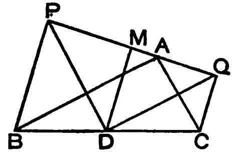
Use Exx. 8 and 9 to show BP ± DR = 2OS, etc.

11. If a straight line be divided into any number of equal parts, and a series of parallel lines be drawn through the points of division, the intercepts made by these lines on any intersecting straight line shall be equal.



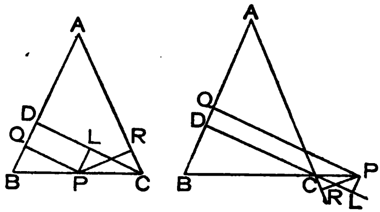
Draw AKL ∥ FJ. Use Ex. 1. to show AK = KL; and hence, by Euc. I. 34, FG = GH, etc.

12. ABC is a triangle, and BP, CQ are drawn perpendicular to PQ, a line through A, and D is the mid-point of BC; show that PD=QD.



Draw DM ⊥ PQ, and use the method of Ex. 11 and Euc. I. 4.

13. If from any point in the base (or base produced) of an isosceles triangle, straight lines be drawn to the sides making equal angles of given magnitude with the base, the sum (or difference) of these lines shall be constant.



Let PQ, PR, CD make equal angles QPB, RPC, DCB, with BC.

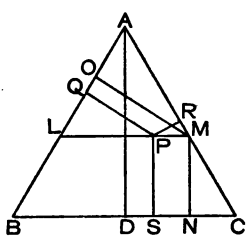
Draw PL || AB.

Use Euc. I. 29, I. 5, and I. 26 to show that

$$CL = PR;$$

and deduce that $PQ \pm PR = CD$.

14. The sum of the perpendiculars drawn from any point within an equilateral triangle to its sides is constant.



Let PQ, PR, PS be the perpendiculars.

Draw LPM || BC; AD and MN || PS; MO || PQ.

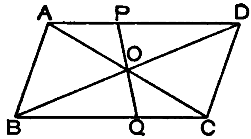
Show as in Ex. 13 that

$$PQ + PR = MO,$$

$$PQ + PR + PS = MO + MN = AD.$$

DEFINITION.—*The point in which the diagonals of a parallelogram intersect is called the* **Centre** *of the parallelogram.*

15. Any straight line drawn through the centre of a parallelogram and produced to meet the sides is bisected by the centre.

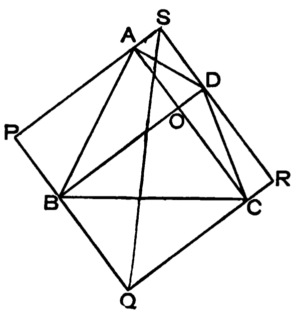


Use § 15, Ex. 1, and the method of § 11, Ex. 1.

16. Any straight line drawn through the centre of a parallelogram bisects the parallelogram.

In above fig. show $\triangle POD \equiv \triangle QOB$, and use Euc. I. 34.

17. The area of any quadrilateral is equal to that of a triangle having two sides and the contained angle equal to the diagonals of the quadrilateral and their contained angle.



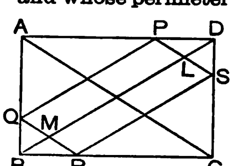
Draw PBQ, SDR ∥ AC, and SAP, RCQ ∥ DB.

Use Euc. I. 34 to show that

$ABCD = \frac{1}{2} \square PQRS = \triangle QRS.$

Cor.—Two quadrilaterals are equal when their diagonals are equal and contain equal angles.

18. ABCD is a rectangle, and P is any point on one of its sides; show that a parallelogram PQRS may be inscribed in the rectangle whose sides are parallel to the diagonals of the rectangle, and whose perimeter is equal to the sum of the diagonals.



Draw PQ ∥ DB, PLS and QMR ∥ AC.
Use Euc. 1-34 to show
PL = QM, PQ = LM
and Euc. I. 29, 6, and 32 to show
PL = LD = LS, etc.

19. ABC is a triangle; and from D, the mid-point of BC, DE and DF are drawn ∥ BA and CA to meet AC and AB in E and F; show that FE ∥ BC. Use Ex. 1.

20. ABCD is a quadrilateral having AB ∥ DC and BC = AB + CD; show that the bisectors of angles B, C intersect in the mid-point of AD.

Cut off from BC, BF = AB, show ∠ AFD a right angle, and use § 1, Ex. 2, and § 16, Ex. 1.

21. If through the vertices of a triangle straight lines be drawn parallel to the opposite sides, these straight lines shall form a triangle equiangular with the given triangle, but of four times its area. Use Euc. I. 34.

22. If one diagonal of a parallelogram be equal to one of its sides, the other diagonal shall be greater than any of its sides.

Use Euc. I. 5 and I. 19.

23. AB is a given straight line, and from A a straight line AC (greater than ½AB) is drawn in any direction and produced its own length to D. DB is joined, and with A as centre and AD as radius a circle is described cutting DB produced in E. From AE AF is cut off equal to AC, and CF is joined. Show that CF bisects AB.

24. If the mid-points of two opposite sides of a parallelogram are also the mid-points of two opposite sides of another parallelogram, then the remaining sides of the two parallelograms are equal and parallel.

Use Euc. I. 33 and I. 34.

25. The sum of the perpendiculars on the sides of a parallelogram from an internal point is constant.

Examine also the case of an external point.

26. E, F are the mid-points of the sides CA, AB of the triangle ABC, and AP is the altitude from A; show that ∠ EPF = ∠ A.

Use § 12, Ex. 1 and Euc. I. 5.

27. The area of a rhombus is equal to half the rectangle contained by its diagonals.

Use the method of Ex. 17.

28. ABCD is a parallelogram and AP, BQ, CR, DS are parallel straight lines, meeting a straight line PS which does not cut the parallelogram; show that AP + CR = BQ + DS.

Find the intersection of the diagonals and use Ex. 8.

29. ABCD is a parallelogram and AP, BQ, CR, DS are parallel straight lines meeting a straight line PS which cuts the parallelogram; show that $AP \pm CR = \pm BQ \pm DS$, the – signs being taken when the points B, C, D are on the opposite side of the line from A.

Use Exx. 8 and 9; compare Ex. 10.

30. The sum of the perpendiculars from the extremities of a diameter of a circle on a fixed straight line which does not cut the circle is constant.

Use Ex. 8. Examine the case when the fixed line cuts the circle.

31. The sum of the distances of the vertices of a quadrilateral from a straight line which does not cut the quadrilateral is four times the distance from the same line of the point of intersection of the lines joining the mid-points of opposite sides of the quadrilateral.

Use Exx. 3 and 8 and § 15, Ex. 1.

32. ABC is a triangle in which $BC > AB$; and from BC, BD is cut off $= \frac{1}{2}(AB+AC)$, and BA is produced to E so that $BE = BD$; show that DE bisects AC.

From BC cut off $BF = AB$. Use Euc. I. 32 and Euc. I. 28 to show $AF \parallel DE$, and apply Ex. 1.

33. ABCD is a quadrilateral, and E, F, G, H are the mid-points of AB, BC, CD, DA; the diagonal BD meets EF in P and GH in R, and the diagonal AC meets FG in Q and HE in S; show that the quadrilateral PQRS is equiangular with the quadrilateral ABCD, and equals a fourth of it.

Use Ex. 1 and 2.

34. P is a point outside an equilateral triangle ABC, and PL, PM, PN are drawn perpendicular to BC, CA, AB; show that if P be within the angle BAC or its vertically opposite angle, $PL \sim (PM + PN)$ is constant.

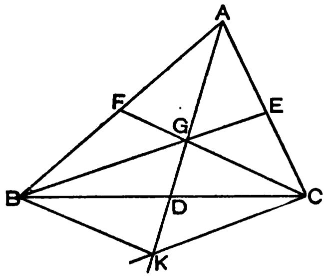
Compare Ex. 14, or use the method of areas. See § 18, Ex. 5. Examine other positions of P.

35. O is any point within ▱ABCD. If ▱s AOBE, BOCF, CODG, DOAH be completed, and EF, etc. joined, prove EFGH a parallelogram and equal to twice ▱ABCD.

§ 17. (*Bookwork*, Euclid, I. 1-39.)

1. The medians of a triangle are concurrent.

(*A Standard Theorem.*)



Let the medians BE, CF meet in G, and let AG produced meet BC in D; it is required to prove that

BD=DC.

Draw CK ∥ EB, meeting AD produced in K. Join BK.

Then in △ ACK, AE=EC and EG ∥ CK;

and a straight line drawn through the mid-point of a side of a triangle parallel to the base bisects the other side;

∴ AG=GK. [§ 16, Ex. 1.

Also in △ ABK, AF=FB and AG=GK;

and the straight line which joins the mid-points of two sides of a triangle is parallel to the base;

∴ FG ∥ BK; [§ 16, Ex. 1.

∴ BGCK is a ▱ and its diagonals bisect each other;

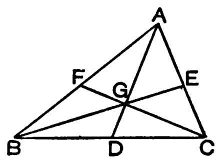
∴ BD=DC. [§ 15, Ex. 1.

Observe that $GD = \frac{1}{2}GK = \frac{1}{2}AG$, or G is a point of trisection of AD. Also $GE = \frac{1}{2}KC = \frac{1}{2}BG$. Similarly $GF = \frac{1}{2}CG$.

Thus the point in which the medians intersect is a point of trisection of each median.

It is shown in Dynamics to be the **Centre of Gravity** *of the triangle.*

2. The sum of the medians of a triangle is greater than three-fourths of its perimeter.



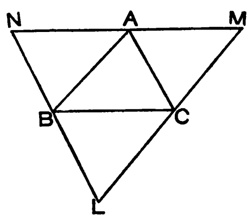
$BG = \frac{2}{3}BE$, etc.; $BG + GC > BC$;

$\therefore \frac{2}{3}(BE + CF) > BC$;

$\therefore BE + CF > \frac{3}{2}BC$.

Add similar results, and divide by 2.

3. If through the angular points of a triangle, straight lines be drawn parallel to the opposite sides, a triangle shall be formed each of whose sides is double the corresponding side of the first triangle.



Draw NAM $\parallel$ BC, etc.

In ▱s ABLC, ABCM,

we have $AB = LC$, $AB = CM$, etc.

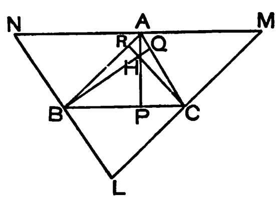
Observe that the area of the

$\triangle LMN = 4 \triangle ABC$,

and that $\angle L = \angle A$, etc.

4. The altitudes of a triangle are concurrent.

(A Standard Theorem.)



Draw NAM $\parallel$ BC, etc.

Then in $\triangle LMN$, show by § 3, Ex. 6 that the lines drawn from

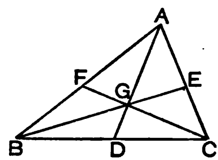
A, B, C $\perp$ MN, NL, LM

(that is to BC, CA, AB)

are concurrent.

DEFINITION.—*The point* H, *in which the altitudes meet, is called the* **Orthocentre** *of the* $\triangle$ ABC.

5. The greater side of a triangle has the less median drawn to it, and conversely.



Let AB > AC.

Use Euc. I. 25 to show that

in △s ADB, ADC, ∠ ADB > ∠ ADC.

Use Euc. I. 24 to show that

in △s GDB, GDC, GB > GC;

∴ GE > GF; and ∴ BE > CF.

The converse is proved by reversing the order of the steps.

6. A triangle is divided by its medians into six triangles which are equal in area.

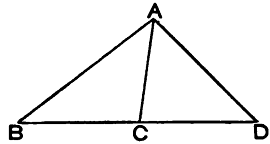
In the figure of Ex. 5, use Euc. I. 38 to show

△ ABD = △ ACD, △ GBD ≡ △ GCD;

∴ △ GAB = △ GCA;

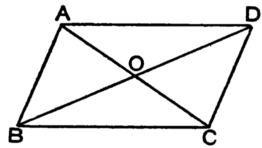
∴ their halves are equal, etc.

7. If two triangles have two sides of the one respectively equal to two sides of the other and the contained angles supplementary, the triangles shall be equal in area.



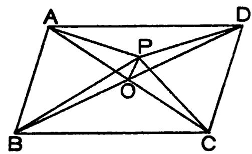
Place the triangles so that two equal sides coincide and the supplementary angles are adjacent. Use Euc. I. 14 and 38.

8. The diagonals of a parallelogram divide it into four equal triangles.



Use Euc. I. 37.

9. O is the centre of the ▱ ABCD, and P is any point within the △ AOD; show that △ BCP = △ APD + △ APC + △ BPD.



Join PO.

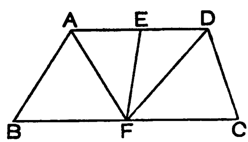
△ BPC = △ POB + △ POC + △ BOC,

but △ BOC = △ AOD = etc.

Examine the cases in which P is (1) outside of the △ AOD but within the ∠ AOD; (2) within the △ AOB, etc.

Definition.—*A quadrilateral which has two sides parallel is called a* **Trapezium.**

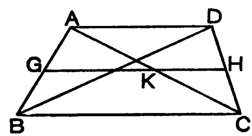
10. The straight line joining the mid-points of the parallel sides of a trapezium bisects the trapezium.



Join AF, DF, and use Euc. I. 38.

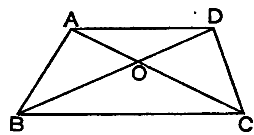
11. If a straight line be drawn through the mid-point of one of the oblique sides of a trapezium parallel to the parallel sides, it shall bisect the opposite side and the diagonals.

Join AC, meeting GH in K. Use § 16, Ex. 1 to show AK=KC, and hence DH=HC. Similarly show that BD is bisected by GH.



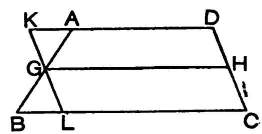
12. O is the point of intersection of the diagonals of the trapezium ABCD (AD ∥ BC); show that △ AOB=△ DOC.

Use Euc. I. 37.



13. The straight line joining the mid-points of the oblique sides of a trapezium is equal to half the sum of the parallel sides.

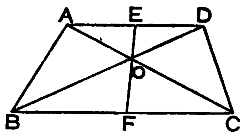
Draw KGL ∥ DC, and show KLCD and KGHD to be ▱s. Prove △ AGK ≡ △ BGL.



14. If through the mid-point of one of the oblique sides of a trapezium a straight line be drawn parallel to the other oblique side, the parallelogram thus formed shall be equal in area to the trapezium.

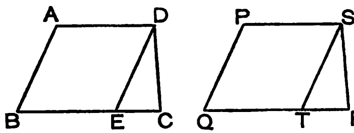
In the figure of Ex. 13, show that △ KGA≡△ LGB, and hence ▱ KLCD=ABCD.

15. The mid-points of the parallel sides of a trapezium and the point of intersection of its diagonals are collinear.



Join EO, OF.
Use Euc. I. 38 to show
△ AEO = △ DEO, etc.
Prove fig. AEOFB = DEOFC,
and use the method of Ex. 10 to
show that EOF coincides with EF.

16. Two trapeziums are congruent if their four sides taken in order are respectively equal.



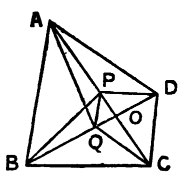
Let ABCD, PQRS be the trapeziums.

Draw DE ∥ AB, ST ∥ PQ.

Show △ DEC ≡ △ STR, etc.

17. O is the point of intersection, and P, Q are the mid-points of the diagonals AC, BD of the quadrilateral ABCD; show that if PQ lie within △ AOB,

(△ AOB + △ COD) − (△ AOD + △ BOC) = 4 △ POQ.



Join AQ, BP, CQ, DP.
Use Euc. I. 38 to show △ APB = △ BPC, etc.
Also △ AOB = △ APB + △ PBQ + △ POQ,
△ COD = △ CPD − △ PDQ + △ POQ,
△ AOD = △ APD + △ PDQ − △ POQ,
△ BOC = △ BPC − △ PBQ − △ POQ, etc.

18. From the mid-points of the diagonals of a quadrilateral lines are drawn parallel to the diagonals. Show that if their point of intersection be joined to the mid-points of the sides, the quadrilateral will be divided into four equal quadrilaterals.

We have to show

LHDG=LGCF

=LFBE=LEAH.

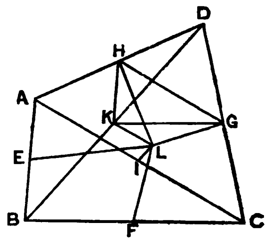
Use Euc. I. 37 to show

LHDG=KHDG

$=\frac{1}{4}\triangle$ ABD$+\frac{1}{4}\triangle$ BCD

[§ 16, Ex. 2.

$=\frac{1}{4}$ABCD.



19. AB, CD are two equal straight lines. If $\angle$ ABC$=\angle$ BCD, and AB be not $\parallel$ CD, then AD $\parallel$ BC.

Use Euc. I. 4 and I. 39.

20. Through the vertices of a $\triangle$ABC parallels are drawn meeting the opposite sides, produced if necessary, in P, Q, R; show that $\triangle$ AQR$=\triangle$ BRP$=\triangle$CPQ$=\triangle$ ABC.

Use Euc. I. 37.

21. ABC is a triangle, and from AC, AP is cut off$=\frac{1}{3}$AC; show that $\triangle$ ABP$=\frac{1}{3}\triangle$ ABC. Use Euc. I. 38.

22. ABCD is a parallelogram. From CB, CE is cut off$=\frac{1}{3}$CB and from CD, CF is cut off$=\frac{1}{3}$CD; show that AECF$=\frac{1}{3}$▱ ABCD.

23. If E and F be the mid-points of the sides CA, AB of the $\triangle$ ABC, and if BE, CF intersect in G, the $\triangle$ GBC=quadrilateral AFGE. Use Euc. I. 38.

24. If in the previous Ex. EF be joined, prove that $\triangle$ AEF $=3\triangle$ EFG.

Use Euc. I. 38, and Ex. 1 (GE$=\frac{1}{2}$BG).

25. In the $\triangle$ ABC, if D be the mid-point of BC and P be any point in AD, $\triangle$ APB$=\triangle$ APC.

26. In the $\triangle$ ABC, D and E are the mid-points of BC, CA, and K is a point in AE. If DK be joined, and AL be drawn $\parallel$ KD, meeting BC in L, show that KL bisects the triangle.

Use Euc. I. 37 and I. 38.

27. In the quadrilateral ABCD, E is the mid-point of BD, and EG $\parallel$ AC meets BC or CD in G. Show that AG bisects the figure.

Use Euc. I. 37 and I. 38.

28. In the quadrilateral ABCD, AC bisects BD. If O is the mid-point of AC, show that $\triangle$ ABO $=\triangle$ BCO $=\triangle$ CDO $=\triangle$ DAO.

29. The base BC of a $\triangle$ ABC is bisected in D, and AD is joined and bisected in E, BE is joined and bisected in F, and CF is joined and bisected in G; show that $\triangle$ EFG $=\frac{1}{8}\triangle$ ABC.

30. P, Q, R are points in the sides BC, CA, AB of the $\triangle$ ABC, such that BP $=\frac{1}{3}$ BC, CQ $=\frac{1}{3}$ CA, AR $=\frac{1}{3}$ AB; show that $\triangle$ PQR $=\frac{1}{3}\triangle$ ABC.

Find the areas of $\triangle$s AQR, etc.

31. ABCD is a square, and AC its diagonal; AD is bisected in E, and BE meets AC in F; show that 12 $\triangle$ AEF $=$ 6 $\triangle$ CEF $=$ 4 $\triangle$ ABE $=$ 3 $\triangle$ BCF.

32. Two points P and Q are taken on AB, one of the parallel sides of the trapezium ABCD, such that AP $=$ BQ, and CP, DQ intersect in O within the figure; show that ADOP $=$ BCOQ in area.

33. The straight line which joins the mid-points of the oblique sides of a trapezium is parallel to the parallel sides.

See Ex. 13.

34. The $\triangle$ ABC when folded over BC comes into the position A′BC; when folded over CA, into the position AB′C; and when folded over AB, into the position ABC′; show that AA′, BB′, CC′ are concurrent.

Use Ex. 4.

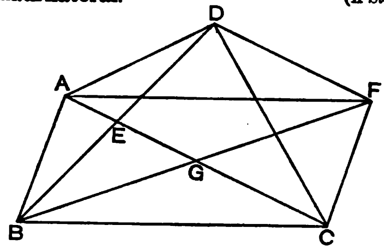
35. ABC is a triangle; BC, CA, AB are produced each twice its own length to D, E, F; show that $\triangle$ DEF $=$ 19 $\triangle$ ABC.

36. In the trapezium ABCD, E, G are the mid-points of the parallel sides AB, DC, and F, H are the mid-points of the oblique sides BC, AD. If HC, FD meet in O, show that

(1) AEOH $=$ BEOF (in area).
(2) EG passes through O.

§ 18. (*Bookwork*, EUCLID, I. 1-41.)

1. If the area of a quadrilateral be bisected by one of its diagonals, this diagonal shall bisect the other diagonal; and conversely, if one diagonal bisect the other diagonal, it shall bisect the quadrilateral. (*A Standard Theorem.*)



Let ABCD be a quadrilateral such that $\triangle ABC = \triangle ADC$, and let AC, BD meet in E; it is required to prove that

BE = ED.

Through A, C draw AF, CF ‖ BC, BA respectively.

Join BF, DF, and let BF meet AC in G.

Then $\triangle AFC = \triangle ABC$; [Euc. I. 34.

but $\triangle ABC = \triangle ADC$; [Hypothesis.

$\therefore \triangle AFC = \triangle ADC$;

$\therefore$ DF ‖ AC. [Euc. I. 39.

ABCF is a ▱, and its diagonals bisect each other;

$\therefore$ BG = GF; [§ 15, Ex. 1.

and the straight line drawn through the mid-point of a side of a triangle parallel to the base bisects the other side;

$\therefore$ BE = ED. [§ 16, Ex. 1.

Conversely, let BE = ED; it is required to prove that

$\triangle ABC = \triangle ADC$.

Complete the ▱ ABCF, and join BF, meeting AC in G.

Then, as before, BG = GF, [§ 15, Ex. 1.

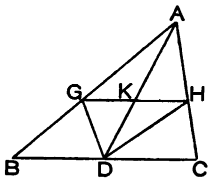
and BE = ED; [Hypothesis.

and the straight line which joins the mid-points of the sides of a triangle is parallel to the base;

$\therefore$ AC ‖ DF; [§ 16, Ex. 1.

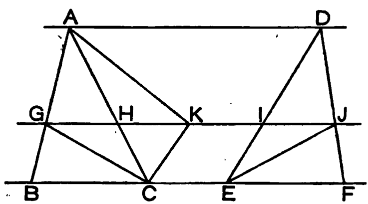
$\therefore \triangle ADC = \triangle AFC = \triangle ABC$. [Euc. I. 37 and 34.

2. A median of a triangle bisects every straight line drawn parallel to the base and terminated by the sides, or sides produced. (*A Standard Theorem.*)



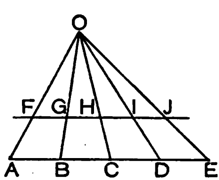
Join GD, DH.
Use Euc. I. 38 to show that
$\triangle ABD = \triangle ADC$,
$\triangle GBD = \triangle HDC$;
$\therefore \triangle AGD = \triangle AHD$,
and apply Ex. 1.

3. If two triangles stand on equal bases in the same straight line, and are between the same parallels, and if a straight line be drawn parallel to the bases, the intercepts made by the sides of the triangles (or the sides produced) shall be equal. (*A Standard Theorem.*)



Let ABC, DEF be given triangles.
Produce GH to K so that $HK = IJ$.
Use Euc. I. 38 to show
$\triangle AKC = \triangle DJE$
$= \triangle AGC$,
and apply Ex. 1.

4. If a given straight line be divided into any number of equal parts, and straight lines be drawn from any given point to the points of section, the intercepts made by these lines on any line parallel to the given line shall be equal.

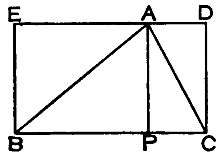


Use the previous exercise to show that any two of the parts into which FJ is divided must be equal.

NOTE.—These theorems (Exx. 1-4) may also be proved indirectly by means of Euc. I. 38, but the direct proof appears preferable.

5. The area of a triangle is one-half that of a rectangle whose base and altitude are equal to those of the triangle.

(A Standard Theorem.)



Draw ED || BC, and BE, CD || PA. Use Euc. I. 41.

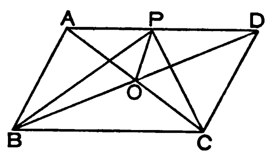
Cor. i.—Triangles on equal bases and of equal altitudes are equal.

Cor. ii.—Equal triangles on equal bases have equal altitudes.

Cor. iii.—Equal triangles with equal altitudes have equal bases.

(Standard Theorems.)

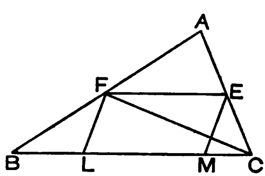
6. O is the centre of the ▱ ABCD, and P is any point in AD; show that PBOC = ½ ▱ ABCD.



Join PO.

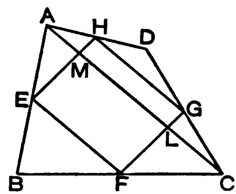
Use Euc. I. 38 to show △ POB = △ POD,
△ POC = △ POA, etc.

7. If the mid-points of two sides of a triangle be joined, and any two parallel lines be drawn from them to the base, the parallelogram thus formed shall be equal to half the triangle.



Join FC. Use Euc. I. 41 and I. 38.

8. The parallelogram formed by joining the mid-points of the adjacent sides of any quadrilateral equals one-half the quadrilateral.

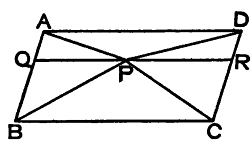


Show as in previous Ex. that

EFLM $= \frac{1}{2} \triangle$ ABC, etc.

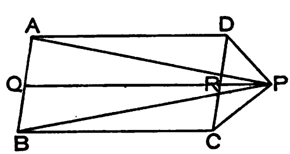
9. P is any point within a ▱ ABCD; show that

$\triangle$ PBC + $\triangle$ PAD = $\frac{1}{2}$ ▱ ABCD.



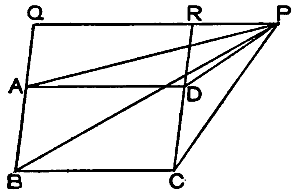
Draw QPR ∥ AD, and use Euc. I. 41.

10. P is any point outside the ▱ ABCD between AD and BC produced; show that $\triangle$ PBC + $\triangle$ PAD = $\frac{1}{2}$ ▱ ABCD.



Draw PRQ ∥ DA, and use Euc. I. 41.

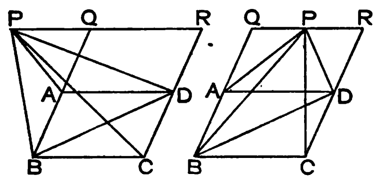
11. P is a point outside the ▱ ABCD on the side of AD remote from BC; show that △ PBC − △ PAD = ½ ▱ ABCD.



Use Euc. I. 41.

Observe that Exx. 9, 10, and 11 are particular cases of one general theorem, and that the sign prefixed to the △ PAD changes when P crosses AD.

12. ABCD is a parallelogram and P is any point; it is required to prove that △ PBD = △ PBC ± △ PAB, according to the position of P.



Produce BA, CD, and draw PQR ∥ AD.

Use Euc. I. 41.

△ PBD = △ PAD + △ ABD ± △ PAB
= ½ ▱ QD + ½ ▱ AC ± △ PAB
= ½ ▱ QC ± △ PAB
= △ PBC ± △ PAB.

Note.—On this proposition depends an important theorem in Dynamics.

13. ABC, DBC are two triangles on the same base BC and between the same parallels AD, BC; a straight line parallel to BC cuts AB in P, AC in Q, DB in R, DC in S; show that PQ = RS.

Use Ex. 1 as in Ex. 3.

14. Two triangles of equal area stand on opposite sides of equal bases in the same straight line; show that the straight line which joins their vertices is bisected by their base (produced if necessary).

Use Ex. 1.

15. If in the quadrilateral ABCD, $\angle$ ADC = $\angle$ BCD, and $\angle$ DAC = $\angle$ CBD, show that AB $\parallel$ DC.

Use Euc. I. 26 and I. 39.

16. The perimeter of an isosceles triangle is greater than the perimeter of an equal rectangle whose altitude is equal to that of the triangle.

Use Euc. I. 41 and I. 19.

17. If ABC be a triangle right-angled at A, and AP be the altitude from A, show that the rectangle AP·BC = rectangle AB·AC.

Use Ex. 5.

18. The sum of the perpendiculars from an internal point on the sides of a convex equilateral polygon is constant.

Join the point to the vertices of the polygon, and use Ex. 5.

19. The algebraical sum of the perpendiculars from any point on the sides of an equilateral polygon is constant, if the perpendiculars be considered negative which fall on the sides, or sides produced, from the outside.

20. AK is the bisector of the $\angle$ A of a $\triangle$ ABC, and meets BC in K. If BD, CE be drawn perpendicular to AE, the bisector of the exterior angles at A, prove that CD and BE pass through the mid-point of AK.

Produce CE and BA to meet, and use Ex. 2.

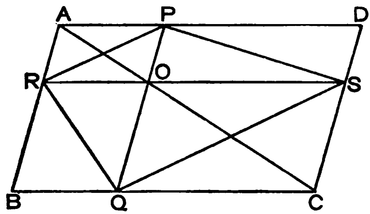
21. If AB be one of the oblique sides of a trapezium ABCD, and E be the mid-point of CD, $\triangle$ ABE = $\frac{1}{2}$ ABCD.

Use the method of § 17, Ex. 14.

22. If of the four triangles into which the diagonals divide a quadrilateral two opposite ones be equal, the figure shall be a trapezium.

§ 19. (*Bookwork*, Euclid, I. 1-43.)

1. A pair of the diagonals of the parallelograms which are about the diagonal of a parallelogram are parallel.



Let ABCD be a ▱, and let PQ, RS be drawn through O, a point in AC, parallel to AB, AD; it is required to prove that PR ∥ SQ.

Join PS, RQ.

▱ BO = ▱ DO; [Euc. I. 43.

$\therefore$ △ROQ = △POS. [Euc. I. 34.

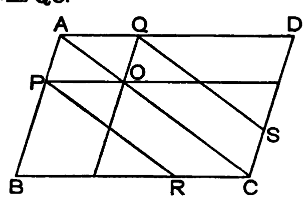
Add to each △QOS;

$\therefore$ △RQS = △PQS;

$\therefore$ RP ∥ QS. [Euc. I. 39.

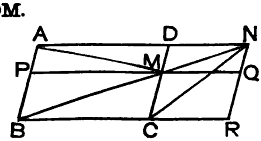
2. ABCD is a parallelogram, and APOQ is a parallelogram about its diagonal; if PR, QS be drawn parallel to AC to meet BC, CD in R, S, show that ▱ PC = ▱ QC.

Use Euc. I. 35 to show ▱ PC = ▱ OB, etc., and apply Euc. I. 43.

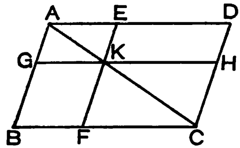


3. ABCD is a ▱, and the straight line BMN meets CD in M and AD in N; show that △CMN = △ADM.

Complete the ▱ CDNR, draw PMQ ∥ AN, and use Euc. I. 43 and I. 41.

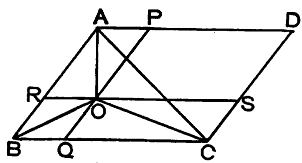


4. EKF and GKH are drawn parallel to the sides AB and AD of ▱ ABCD, so that ▱ EH = ▱ GF; show that AKC is a straight line.



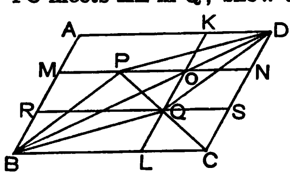
Show that the various parts of the figure AKCD = the corresponding parts of AKCB, and apply Euc. I. 34.

5. ABCD is a parallelogram, O is any point within △ ABC, and POQ, ROS are parallel to AB, BC respectively; show that ▱ PS − ▱ RQ = 2 △ AOC.



$$2\triangle AOC = 2(\triangle ABC - \triangle AOB - \triangle BOC)$$
$$= ▱ ABCD - ▱ ABQP - ▱ RBCS, \text{ etc.}$$

6. ABCD is a parallelogram, and O is any point on BD; KOL, MON are parallel to AB, BC; P is any point in MN, and PC meets KL in Q; show that PD ∥ BQ.



Join PB, DQ, draw RQS ∥ AD, prove △ DBQ = △ PBQ, and apply Euc. I. 39.

$$\triangle PBQ = \triangle PBC - \triangle QBC = \tfrac{1}{2} ▱ MS.$$
$$\triangle DBQ = \triangle BCD - \triangle BCQ - \triangle CDQ, \text{ etc.}$$

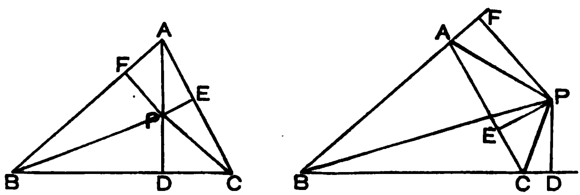
7. The diagonals of the parallelograms which are about the diagonal of any parallelogram are parallel to those of the parallelogram. Use Euc. I. 43, 41, and 39.

8. P is a point on the diagonal AC of a ▱ ABCD, or on AC produced; show that if PB, PD be joined, the parallelogram is divided into two pairs of equal triangles.

§ 20. (*Bookwork*, EUCLID, I. 1-48.)

1. ABC is a triangle, and P is any point; PD, PE, PF are perpendicular to BC, CA, AB; it is required to show that

$$AF^2+BD^2+CE^2=AE^2+BF^2+CD^2.$$



Join PA, PB, PC.

$PA^2=PF^2+AF^2.$ [Euc. I. 47.

Similarly $PB^2=PD^2+BD^2,$

$PC^2=PE^2+CE^2;$

$\therefore\ PA^2+PB^2+PC^2=(PD^2+PE^2+PF^2)+(AF^2+BD^2+CE^2);$

Again $PA^2=PE^2+AE^2.$ [Euc. I. 47.

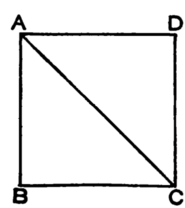
Similarly $PB^2=PF^2+BF^2,$

$PC^2=PD^2+CD^2;$

$\therefore\ PA^2+PB^2+PC^2=(PD^2+PE^2+PF^2)+(AE^2+BF^2+CD^2);$

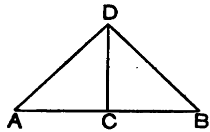
$\therefore\ AF^2+BD^2+CE^2=AE^2+BF^2+CD^2.$

2. ABCD is a square; show that $AC^2=2\,AB^2$.



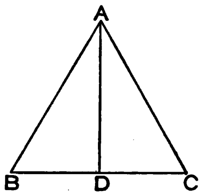
Use Euc. I. 47.

3. AB is bisected in C; show that $AB^2=4AC^2$.



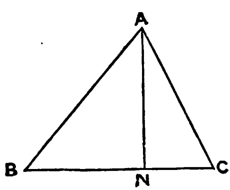
Draw $CD \perp AB$, and $= AC$.
Join AD, BD.
Use Euc. I. 47, to show that
$AB^2=2AD^2=4AC^2$.

4. ABC is an equilateral triangle and AD is perpendicular to BC; show that $AD^2=3BD^2$.



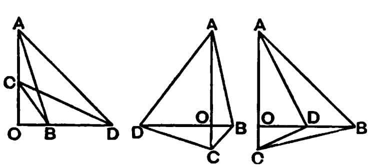
Prove that BC is bisected in D, and use Euc. I. 47.

5. If AN be the altitude from A in the $\triangle ABC$; show that $AB^2 \sim AC^2=BN^2 \sim CN^2$.



Use Euc. I. 47.

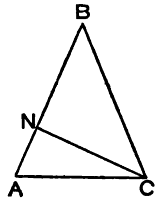
6. AOB is a right angle, C is any point in AO, or AO produced, D is any point in BO or BO produced; show that $AB^2+CD^2=AD^2+BC^2$.



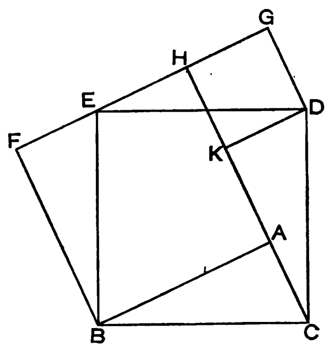
Use Euc. I. 47.

7. In the $\triangle$ ABC, AB=BC, and CN is the altitude from C; show that $BC^2+CA^2+AB^2=AN^2+2BN^2+3CN^2$.

Use Euc. I. 47.



8. Prove Euc. I. 47 by means of a figure in which the square on the hypotenuse is described on the same side of the hypotenuse as the triangle.



Let ABC be the $\triangle$, and BCDE the square on the hypotenuse. Describe $\triangle$s BFE, EGD, CKD, congruent with BAC. Produce CK to H.

Show that FA is the square on BA, and HD is the square on KD or AC, and show figure

FGDKAB=BCDE.

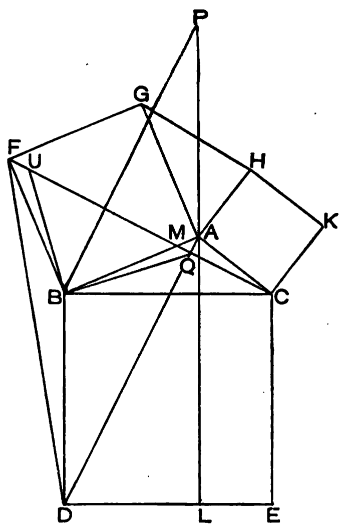
9. If ABC be any triangle, and squares BCED, ABFG, ACKH be described on its sides, prove the following properties :—

(1) FC ⊥ AD and BK ⊥ AE.

(2) If FC, AD meet in Q, BQ bisects ∠ FQD.

(3) △ ABC = △ AGH = △ BDF = △ CEK.

(4) AL, BK, CF are concurrent.



(1) Show △ FBC ≡ △ ABD (Euc. I. 4) and △ FMB equiangular to △ AMQ.

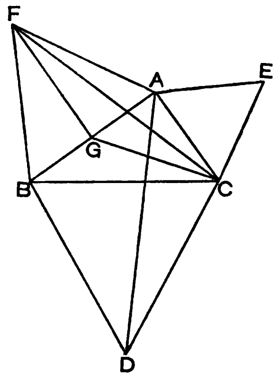
(2) Cut off FU = AQ, and show △ UBC ≡ △ QBD, and ∠ BUQ = ∠ BQU = ∠ BQD.

(3) Use the method of § 17, Ex. 7.

(4) Produce LA to P so that AP = DB, and apply § 17, Ex. 4 to △ PBC.

10. ABC is a triangle right-angled at A; on BC, CA, AB equilateral triangles DBC, ECA, FAB are described; show that

$\triangle DBC = \triangle ECA + \triangle FAB$.



Draw FG $\perp$ AB. Join AD, CF, CG.

Use Euc. I. 4 to show $\triangle ABD = \triangle FBC$.

But $\triangle FBC = \triangle BGC + \triangle FBG + \triangle FGC$.

$= \frac{1}{2}\triangle ABC + \triangle FAB$. [Euc. I. 37.

Similarly, etc.

11. If ABC be a triangle right-angled at A, and squares BCED, ABFG, ACKH be described on its sides, prove that (in addition to the properties proved in Ex. 9)—

(1) BG $\parallel$ CH.

(2) Points F, A, K are collinear.

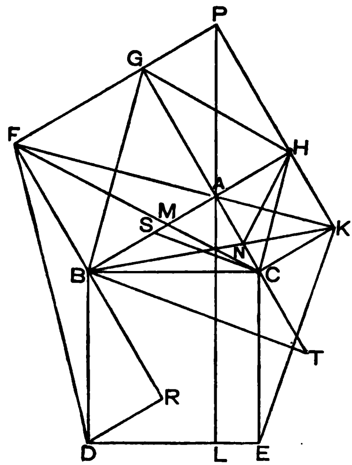
(3) $FD^2 = 4AB^2 + AC^2$; $EK^2 = 4AC^2 + AB^2$.

(4) $FD^2 + EK^2 = 5BC^2$.

(5) FG, HK, AL are concurrent.

(6) If FC meet AB in M, and BK meet AC in N, AM=AN.

(7) The medians of △ABC are perpendicular to the lines GH, FD, EK, and equal to their respective halves.



(1) Use Euc. I. 39.

(2) Use Euc. I. 14.

(3) Produce FB to R, draw DR⊥BR,
prove △ RBD ≡ △ ABC, and use I. 47.

(4) This follows from (3) and Euc. I. 47.

(5) Let FG, KH meet in P, and prove △ GAP≡△ ABC, and PA, AL in the same straight line.

(6) Use Euc. I. 37 to show
△ BNH=△ ABK=△ ABC=△ GMC,
and apply § 18, Ex. 5, Cor. ii.

(7) Produce AC its own length to T, and show CS=and ∥ ½BT. Also show △KCE≡△TCB and KE⊥BT.

12. On the sides AB, BC, of the triangle ABC, any parallelograms ADEB, BFGC are described, and DE, GF, produced if necessary, meet in H; show that a parallelogram described on AC, with two sides equal and parallel to BH, is equal to the sum of the parallelograms on AB and BC.

Use Euc. I. 35.

13. From the preceding Exercise obtain a proof of Euclid I. 47.

14. If ABCD be a rectangle and O be any point, then

$$OA^2+OC^2=OB^2+OD^2.$$

Use Euc. I. 47.

15. In the $\triangle$ ABC the angles A, C are each half a right angle; AD bisects $\angle$ A and meets BC in D; show that $DC^2=2BD^2$.

Draw DF $\perp$ AC, and use Euc. I. 26 and I. 47.

16. Let A and B be two fixed points, and let LM be a straight line $\perp$ AB. If AB, produced if necessary, meet LM in O, and P be any point in LM, then $PA^2+BO^2=PB^2+AO^2$.

17. Show that the sum of the squares on the lines joining the angular points of a rectangle to any point within it is double of the squares of the perpendiculars from that point on the sides.

18. In the figure of Ex. 11 show that the sum of the squares of the sides of the figure FGHKED is $8BC^2$.

See Ex. 11 (3).

19. In any triangle the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle. Use Euc. I. 47 and I. 24.

20. In an obtuse-angled triangle the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing that angle.

21. Enunciate and prove the converses of the two preceding theorems.

22. ABC . . . is a polygon, and from P, a point within it, PL is drawn $\perp$ AB, PM $\perp$ BC, etc.; show that

$$AL^2+BM^2+\ldots=BL^2+CM^2+\ldots$$

23. Two triangles, ABC, PQR, have their sides respectively parallel; QA_1 and RA_2 are drawn $\perp$ BC, RB_1 and $PB_2 \perp$ CA, PC_1 and $QC_2 \perp$ AB; show that

$$PA_1{}^2+QB_1{}^2+RC_1{}^2=PA_2{}^2+QB_2{}^2+RC_2{}^2.$$

Draw the altitudes of $\triangle$ PQR, and use Euc. I. 47.

§ 21. *Maxima and Minima.*

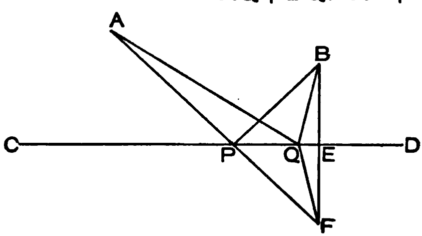
This section contains theorems in Maxima and Minima, and has been placed last on account of the nature of the subject, although the propositions required are in the earlier part of Book I.

1. If, from two fixed points on the same side of a given straight line straight lines be drawn to a point in the given line, the sum of the lengths of these lines shall be least when they make equal angles with the given line.

(*A Standard Theorem.*)

Let A and B be the given points, CD the given straight line; then if P be a point in CD such that $\angle$ APC $= \angle$ BPD, and Q be any other point in CD, it is required to prove that

$$AQ+BQ>AP+BP$$



From B draw BE $\perp$ CD, and produce BE and AP to meet in F.

Join FQ.

Then $\angle$ APC $= \angle$ FPE; [Euc. I. 15.

$\therefore \angle$ BPE $= \angle$ FPE.

In $\triangle$s BPE, FPE $\begin{cases} EP=EP, & \\ \angle BEP=\angle FEP, & \text{[Right angles.} \\ \angle BPE=\angle FPE; & \text{[Proved above.} \end{cases}$

$\therefore \triangle BPE \equiv \triangle FPE$; [Euc. I. 26.

$\therefore$ PB $=$ PF, and BE $=$ FE.

In $\triangle$s BEQ, FEQ $\begin{cases} BE=FE, \\ EQ=EQ, \\ \angle BEQ=\angle FEQ; \end{cases}$

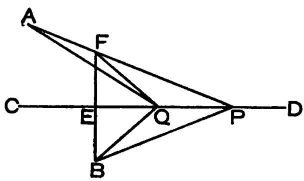
$\therefore \triangle BEQ \equiv \triangle FEQ$; $\therefore$ QB $=$ QF; [Euc. I. 4.

But, AQ$+$QF$>$AP$+$PF; [Euc. I. 20.

$\therefore$ AQ$+$QB$>$AP$+$PB.

NOTE.—It is shown in Optics that rays of light, incident from B on a plane mirror through CD at right angles to the plane of the paper, proceed after reflection as if they came from F. On this account F is sometimes called the *Image* of B in CD.

2. If, from two fixed points on opposite sides of a given straight line straight lines be drawn to a point in the line, the difference of the lengths of these lines shall be greatest when they make equal angles with the given line.



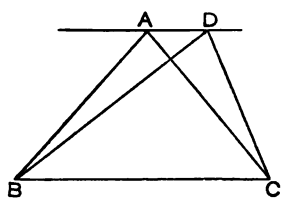
Let $\angle$ APC $= \angle$ BPC.

Draw BE $\perp$ CD, and produce it to meet AP in F.

Show as in § 6, Ex. 7 that AQ $-$ QF $<$ AF,

or, AQ $-$ QB $<$ AP $-$ PB.

3. Of all triangles on the same base and having the same area, the isosceles has the least perimeter.



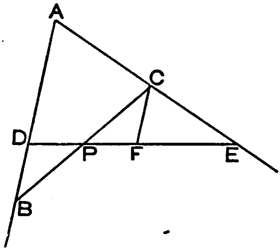
Let AB $=$ AC and AD $\parallel$ BC.

Then $\triangle$ DBC $= \triangle$ ABC. [Euc. I. 37.

Show that AB, AC make equal angles with AD,

and apply Ex. 1.

4. Of all triangles which have the same vertical angle, and whose bases pass through a fixed point between the arms of the angle, the least is that whose base is bisected at the fixed point.

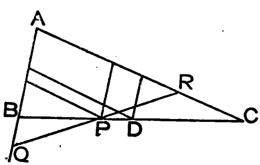


Let P be the fixed point, and let BC be bisected at P, and DPE be any other base.

Draw CF ‖ DB,

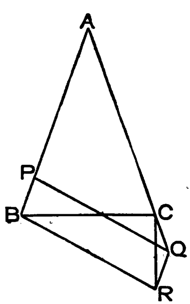
and use Euc. I. 26, etc.

5. From a point in the base of a triangle lines are drawn parallel to the sides. Show that the parallelogram which is thus formed is greatest when the point bisects the base.



Let BC be bisected in D.
Use § 16, Ex. 2 to show
▱ AD $= \frac{1}{2}\triangle$ ABC.
If QPR be drawn through P,
so that QP $=$ PR,
show ▱ AP $= \frac{1}{2}\triangle$ AQR;
and show as in Ex. 4 that
$\triangle$ AQR $<$ $\triangle$ ABC,
and $\therefore$ ▱ AP $<$ ▱ AD.

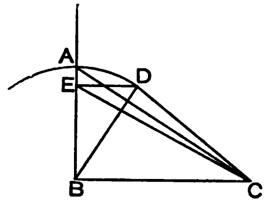
6. Of all triangles having the vertical angle and the sum of the sides the same, the isosceles triangle has the smallest base.



Let ABC be an isosceles $\triangle$, having
AB $=$ AC.
Let AP $+$ AQ $=$ AB $+$ AC.
Draw QR $=$ and ‖ PB.
Show that CQR is isosceles and CR $\perp$ BC,
and $\therefore$ BR (or PQ) $>$ BC.

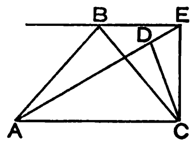
7. Of all triangles having two given sides, the greatest is that in which the contained angle is a right angle.

Let AB, BC be the given sides, and let ABC be a right $\angle$.
Let $DB = AB$.
Draw $DE \perp AB$.
Show that $BE < BD$ and
$\therefore BE < AB$.
Show $\triangle DBC = \triangle EBC < \triangle ABC$.



8. Of all triangles having the same base and the same perimeter, the isosceles has the greatest area.

Let $AB = BC$,
and $AB + BC = AD + DC$.
Let BE be drawn $\parallel$ AC meeting AD in E.
Then $AE + EC > AB + BC$ [Ex. 1.
$> AD + DC$;
$\therefore$ D is between A and E.
Hence show $\triangle ABC > \triangle ADC$.



9. If the diagonals of a quadrilateral be given in magnitude, its area shall be a maximum when they are at right angles.

Use the methods of Ex. 7 and § 16, Ex. 17.

10. If the diagonals of a parallelogram be given, its area shall be a maximum when it is a rhombus.

11. If two sides of a triangle be unequal, the altitude drawn to the shorter side shall be greater than that drawn to the longer.

Use § 18, Ex. 5.

12. XOY is an acute angle, and A, B are points within it. If P, Q be points on OX, OY, show that $AP+PQ+QB$ is a minimum when $\angle APX = \angle OPQ$ and $\angle PQO = \angle BQY$.

Use Ex. 1.

13. If, in the figures of Exx. 1 and 2, AB meet CD in R, show that $AR-BR$ and $AR+BR$ are respectively a maximum and a minimum.

CHAPTER II.

PROBLEMS.

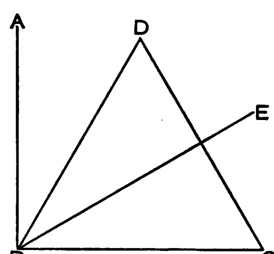
THE solution of problems often presents more difficulty than the proof of theorems, as the enunciation may give little or no clue to the construction required. It will, however, be found that nearly all problems may be solved by one or other of the methods which are given in the following sections, and that many problems may be solved by more than one of these methods. The special attention of the student should be given to the method of *Analysis and Synthesis* described in § 27.

The remarks in *Italics*, which in the illustrative examples follow the general enunciation, are intended to explain to the student the nature of the method employed in the section, and do not require to be written out as part of the solution.

§ 22. *Problems which follow directly from known Theorems.*

1. Trisect a right angle.

We know by Euc. I. 32 *that each angle of an equilateral triangle is two-thirds of a right angle. This suggests the construction which follows.*



Let ABC be a right angle; it is required to trisect it.

Construction—

On BC construct an equilateral △ DBC. [Euc. I. 1.
Through B draw BE bisecting the ∠ DBC. [Euc. I. 9.

∠ ABC shall be trisected by BD and BE.

Proof—

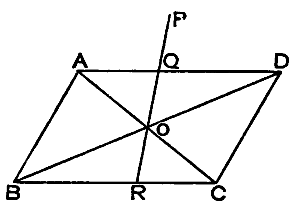
The sum of the three angles of △DBC is 2 rt. ∠ s. [Euc. I. 32.

∴ ∠ DBC is two-thirds of a rt. ∠ ;

∴ each of the ∠ s ABD, DBE, EBC is one-third of a rt. ∠ .

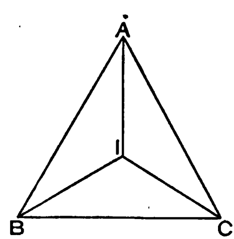
2. Bisect a parallelogram by a straight line drawn through any given point.

See § 16, Ex. 16.

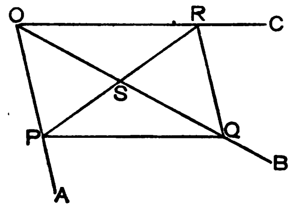


3. Divide an equilateral triangle into three congruent triangles.

Use § 9, Ex. 6, and Euc. I. 26.

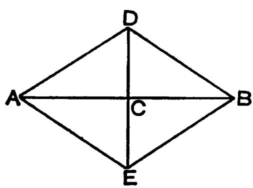


4. OA, OB, OC are three straight lines, and P is any point in OA; show how to draw a straight line through P meeting OB, OC in S, R, so that PS=SR.



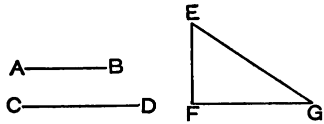
Draw PQ ∥ OC, QR ∥ AO, and join PR.
Use § 15, Ex. 1.

5. Construct a rhombus, having given its diagonals.



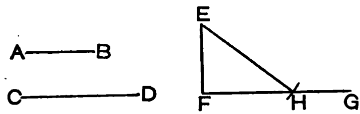
See § 3, Ex. 4.

6. Construct a square equal to the sum of two given squares.



Let AB, CD be sides of the given squares.
Draw EF=AB, FG ⊥ EF and =CD.
Use Euc. I. 47.

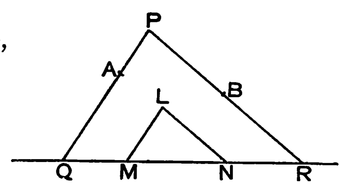
7. Construct a square equal to the difference of two given squares.



Let AB, CD be sides of given squares.
Draw EF=AB, FG⊥EF;
With centre E, and radius equal to CD,
describe a circle cutting FG in H.

8. Through two points draw two straight lines which shall make with a given straight line a triangle equiangular to a given triangle.

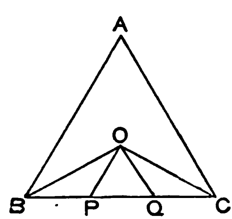
Let LMN ≡ the given triangle,
A, B the given points,
and MN the given line.
Draw PAQ ∥ LM, PBR ∥ LN.



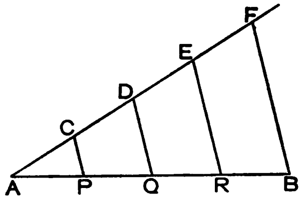
9. Trisect a given straight line.

See § 13, Ex. 4.

Find an alternative method.

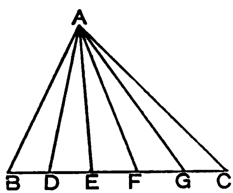


10. Divide a given straight line into any number of equal parts.



Let AB be the given line.
Draw AC, and cut off equal parts AC, CD, etc.
Join FB, and draw parallels.
Then AP=PQ=etc.
See § 16, Ex. 11.

11. Divide a triangle into any number of equal parts by lines drawn through the vertex.



Divide the base into the given number of parts as in Ex. 10.
Join AD, etc.,
and apply Euc. I. 38.

12. Construct an isosceles triangle on a given base, having the angle at the vertex double each of the angles at the base.

Use Euc. I. 32.

13. Construct an isosceles triangle on a given base, having the vertical angle four times each of the angles at the base.

Use Ex. 1.

14. Construct an isosceles triangle on a given base, such that the two angles at the base are together equal to three times the vertical angle.

15. Construct an equilateral triangle with a given straight line as altitude. Use Ex. 1.

16. Bisect a parallelogram by a straight line drawn parallel to a given straight line.

See § 16, Ex. 16.

17. Draw a straight line through a given point so as to make equal angles with two given straight lines which cannot be produced to meet.

Draw parallels through the given point, etc.

18. Divide an equilateral triangle into nine equal triangles whose common vertex is a point within the triangle.

See figure of § 13, Ex. 4.

19. Construct a square, having given its diagonal.

Compare Ex. 5.

20. Construct a square equal to the sum of three or more given squares. See Ex. 6.

21. Construct a square equal to twice, thrice, four times, etc., the area of a given square.

22. Through two given points draw straight lines which shall make an equilateral triangle with a given straight line.

Use the method of Ex. 8.

23. Divide a parallelogram into any number of equal parallelograms by lines drawn parallel to one pair of sides.

See Ex. 10, and use Euc. I. 36.

24. Trisect a parallelogram by lines drawn through an angular point.

See § 17, Ex. 22.

25. On a given straight line describe a rhombus having an angle equal to a given rectilineal angle.

Use the method of Euc. I. 46.

26. On a given straight line describe a rhombus having each of one pair of opposite angles equal to twice each of the other pair.

27. Construct a rectangle equal to a given square, and having one side equal to a given straight line.

Use Euc. I. 43, as in I. 44.

28. Trisect a triangle by straight lines drawn from the angular points to meet at a point within the triangle.

See § 17, Ex. 6.

29. Draw straight lines through the angular points of a quadrilateral so as to form a parallelogram whose area is double that of the quadrilateral.

See § 16, Ex. 17.

Show that an infinite number of solutions exist.

§ 23. *Loci.*

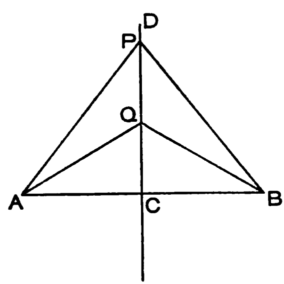
A simple class of problems is that in which it is required to find a *locus.*

DEFINITION.—*The* **Locus** *of a point is the aggregate of all the positions of the point which satisfy a given condition.*

In plane geometry such an aggregate will be found to consist of one or more straight or curved lines or portions of lines.

In order to show that any line (or lines) constitutes the required locus, it is necessary to prove (1) that any point which satisfies the given condition is in the line; (2) that every point in it satisfies the condition.

1. Find the locus of a point which is equidistant from two given points. *(A Standard Locus.)*



Let A, B be the given points.

Join AB, and bisect it in C. [Euc. I. 10.

Then C is equidistant from A and B, and is therefore a point in the required locus.

Proof—

(1) Let P be any other point which satisfies the condition.

Join PC.

In $\triangle$s PAC, PBC $\left\{\begin{array}{l} PC=PC. \\ CA=CB, \\ PA=PB; \end{array}\right.$ [Construction. [Hypothesis.

$\therefore \triangle PAC \equiv \triangle PBC$; [Euc. I. 4.

$\therefore \angle PCA = \angle PCB =$ a rt. $\angle$. [Def. of rt. $\angle$.

It follows that any point, which satisfies the given condition, is in the straight line CD drawn at right angles to AB through its mid-point.

(2) Every point in CD satisfies the condition.

Let Q be a point in CD.

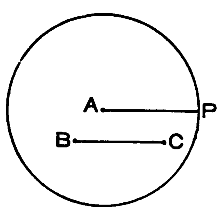
Join QA, QB.

In $\triangle$s QCA, QCB $\left\{\begin{array}{l} QC=QC, \\ CA=CB, \\ \angle QCA = \angle QCB; \end{array}\right.$ [Right angles.

$\therefore \triangle QCA \equiv \triangle QCB$; [Euc. I. 4.

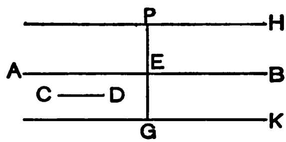
$\therefore QA=QB$.

2. Find the locus of a point whose distance from a given point is equal to a given straight line. (*A Standard Locus.*)



Describe a circle with the given point as centre and the given line as radius, and use the definition of a circle.

3. Find the locus of a point whose distance from a given straight line of unlimited length is equal to a given finite straight line. (*A Standard Locus.*)



We have PE ⊥ AB and = CD.

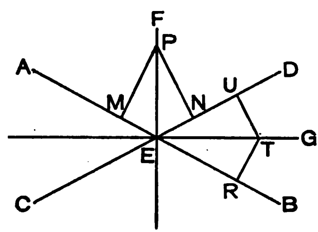
Produce PE its own length to G.

Draw PH, GK ⊥ PG.

PH and GK produced both ways form the locus.

Prove this by means of Euc. I. 34.

4. Find the locus of a point which is equidistant from two intersecting straight lines. (*A Standard Locus.*)



Let AB, CD be the given lines,

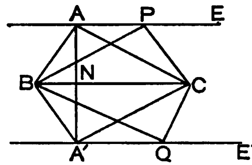
P a point which satisfies the condition.

Join PE.

(1) Prove that PE bisects ∠ AED or ∠ BED.

(2) Prove that any point T on either bisector satisfies the condition.

5. Find the locus of the vertex of a triangle of given area and on a given base. (*A Standard Locus.*)



Let ABC be a triangle of given area on given base BC, and P, Q points which satisfy the condition.

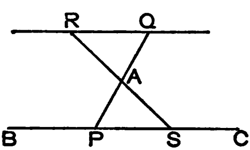
Draw AN $\perp$ BC, and produce it to A′, so that NA′=AN.

(1) Prove PA $\parallel$ BC.

(2) Prove that any point on AP or A′Q satisfies the condition.

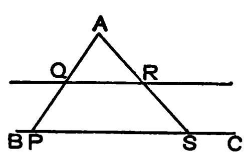
6. A is a fixed point, and BC a fixed straight line; P is any point in BC; PA is joined and produced to Q so that AQ=PA; find the locus of Q.

Through Q draw QR $\parallel$ BC, and use Euc. I. 26 to show that it is the required locus.



7. A is a fixed point, and BC a fixed straight line; P is any point in BC; AP is joined and bisected in Q; find the locus of Q.

Through Q draw QR $\parallel$ BC, and use § 16, Ex. 1.



8. Find the locus of a point whose distance from the circumference of a given circle measured along the radius, or the radius produced, is constant.

Use Ex. 2.

9. AB is a *finite* straight line; find the outline of the locus of a point whose shortest distance from some point in AB is always equal to a given straight line.

Use Exx. 2 and 3.

10. Find the locus of a point which is equidistant from two parallel straight lines.

11. APQR is a rhombus, such that P and R lie on two fixed straight lines through A; find the locus of Q.

See § 3, Ex. 3.

12. APQB is a parallelogram of constant area on a given base AB; find the loci of P and Q.

Use Ex. 5.

13. A is a fixed point, and BC a fixed straight line; P is any point in BC; if AP is produced to Q so that PQ=AP, find the locus of Q.

Use the method of Ex. 6.

14. A is a fixed point, and BC a fixed straight line; P is any point in BC. If PA be produced to Q so that AQ=2PA, find the locus of Q.

15. A is a fixed point, BC a fixed straight line, and n is a whole number; P is any point in BC, and in AP or PA produced a point Q is taken so that AQ=nAP; find the locus of Q in either case.

16. A is a fixed point, BC a fixed straight line, and n is a whole number; P is any point in BC, and in AP or PA produced a point Q is taken such that AP=nAQ; find the locus of Q in either case.

17. A is a fixed point, and P is a point on a circle of which A is centre; find the locus of the mid-point of AP.

18. A is a fixed point, and P is a point on a circle of which B is centre; show that the locus of the mid-point of AP is a circle whose centre is the mid-point of AB and whose radius is ½BP.

Consider the cases in which A is within, on, or outside the given circle.

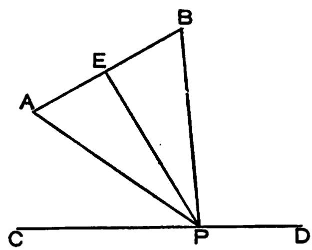
§ 24. *Intersection of Loci.*

In many problems we are required to find the position of a point which satisfies two given conditions. Each of the conditions determines a locus on which the point must lie, and the solution is therefore the point or points common to both loci.

This method of the *Intersection of Loci* is used by Euclid in the 1st and 22d Propositions of Book I.

1. To find a point in a given straight line which shall be equidistant from two given points.

The point required must lie on the locus found in § 23, Ex. 1, *and also on the given line. It must therefore be their point of intersection. Hence we obtain the following construction.*



Construction—

Let A, B be the given points, CD the given straight line.

Join AB. Bisect it in E. [Euc. I. 10.

Draw EP $\perp$ AB, meeting CD in P. [Euc. I. 11.

P shall be the required point.

Proof—

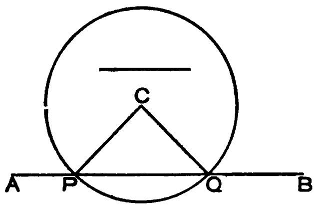
Join AP, BP.

In $\triangle$s AEP, BEP $\begin{cases} AE = BE, & \text{[Construction.} \\ EP = EP, & \\ \angle AEP = \angle BEP; & \text{[Right angles.} \end{cases}$

$\therefore$ AP = BP. [Euc. I. 4.

Note that if AB $\perp$ CD, EP will not meet CD, and there is therefore no solution unless CD pass through E.

2. On a given line of unlimited length find a point whose distance from a given point is equal to a given finite straight line.

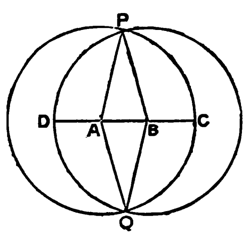


Describe a circle, with the fixed point C as centre, and the given finite line as radius.

Show that either of the points in which it cuts the given line AB satisfies the problem.

Examine the condition that no solution may exist.

3. To describe an isosceles triangle, each of whose sides shall be double the base.

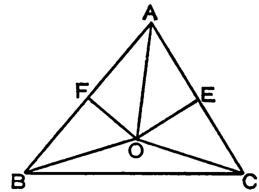


Produce AB its own length both ways to C and D.

With A and B as centres, describe circles intersecting in P and Q.

Show that either PAB or QAB is the triangle required.

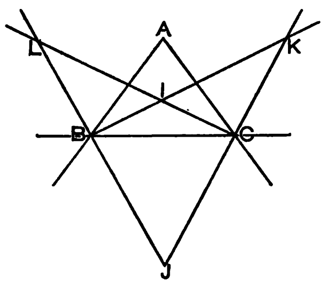
4. Find a point which is equidistant from three given points.



See § 1, Ex. 5.

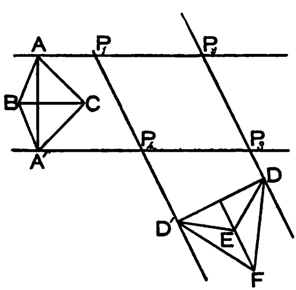
Find in what case this problem has no solution.

5. Find a point which is equidistant from three given lines.



See § 7, Exx. 7 and 8, and show that four points I, J, K, L satisfy the condition. Find in what cases the condition is satisfied by (1) two points, (2) no point.

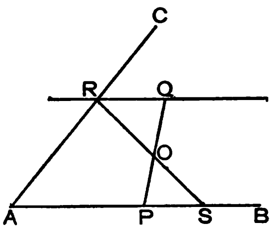
6. ABC, DEF are two triangles; find a point P such that △PBC = △ABC in area, and △PEF = △DEF.



Use § 23, Ex. 5.

Each of the four points P_1, P_2, P_3, P_4 satisfies the conditions.

7. Draw a straight line through a given point such that the part intercepted between two given intersecting straight lines shall be bisected at that point.



Let O be the given point.

Use the method of § 23, Ex. 6 to find the locus of Q, when

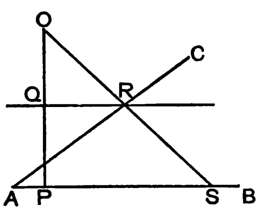
QO=OP.

Let this locus meet AC in R.

ROS shall be the line required.

Prove by Euc. I. 26.

8. O is a given point, and AB, AC are two given straight lines; draw a straight line from O to AB which shall be bisected by AC.



Let P be any point in AB.

Join OP. Bisect it in Q,

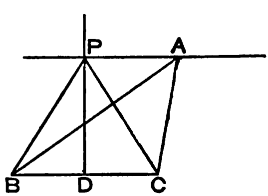
and draw QR || AB.

Join OR, and produce to meet AB in S.

Prove by § 16, Ex. 1.

Compare the method of § 22, Ex. 4.

9. Construct an isosceles triangle equal to a given triangle, and standing on the same base.



Let ABC be given triangle.

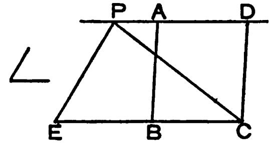
Bisect BC in D.

Draw DP⊥BC and AP || BC.

Use § 1, Ex. 4, and Euc. I. 37.

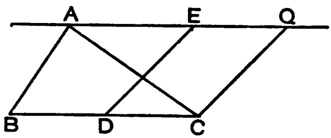
10. Construct a triangle equal to a given parallelogram, and having an angle equal to a given angle.

Let ABCD be the ▱.
Produce CB to E, BE=BC, etc.
Use Euc. I. 41 and I. 37.



11. Construct a parallelogram equal to a given triangle, and having its perimeter equal to that of the triangle.

Bisect BC in D.
Find DE $=\frac{1}{2}$(AB+AC).
Use Euc. I. 31 and I. 41.



12. Given two points and a line, describe a circle such that its centre shall be on the line, and its circumference shall pass through the two points.

See Ex. 1.

13. Find a point at a given distance from a given point, and equally distant from two other given points.

Use § 23, Exx. 1 and 2.

14. Find a point at a given distance from a given point and at a given distance from a given straight line.

Use § 23, Exx. 2 and 3. Show under what circumstances there may be 4, 3, 2, 1, or no solutions.

15. Find a point whose distances from two given straight lines are respectively equal to two given finite straight lines.

Use § 23, Ex. 3. Compare § 24, Ex 6.

16. Find a point whose distances from two given circles measured along the radii or radii produced are respectively equal to two given straight lines.

17. If *n* be a whole number, show how to describe an isosceles triangle each of whose sides shall be *n* times its base.

Use the method of Ex. 3.

18. Describe an isosceles triangle on a given base with sides equal to a given straight line.

19. Find a point in the base of a triangle, or the base produced, equidistant from the two sides.

Use § 23, Ex. 4, and show that in general either of two points satisfies the conditions. When will there be only one point?

20. Draw a line through a given point such that the given point shall be a point of trisection of the part intercepted between two given lines.

Employ the loci found in § 23, Exx. 14 and 16.

21. O is a given point, and AB, AC are two given straight lines. Draw a straight line from O to AC, such that AB shall cut it in a point of trisection.

22. O is a given point, AB, AC are two given straight lines, and *n* is a given whole number. Draw a straight line POQ meeting AB in P, AC in Q, so that OP = *n* times OQ.

Employ the locus found in § 23, Ex. 16.

23. O is a point surrounded by a closed curve ABC, and DE is a straight line outside the curve. Draw a straight line from O to meet ABC in P and DE in Q, so that OP = PQ.

Use the method of Ex. 8.

24. Construct a right-angled isosceles triangle equal to a given square.

Compare Ex. 10.

25. Construct a parallelogram equal to a given square standing on the same base and having an angle equal to half a right angle.

26. Construct a rhombus equal to a given parallelogram having each of its sides equal to the longer side of the parallelogram.

Use Euc. I. 35.

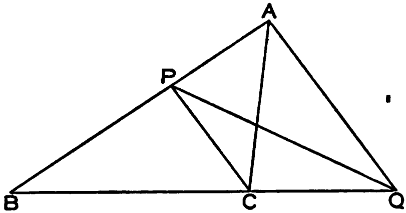
27. Construct a rhombus equal to a given triangle, having each of its sides equal to the base of the triangle.

Use the method of the previous Ex., making the altitude of the rhombus half that of the triangle.

§ 25. *Intersection of Loci—(continued).*

A number of similar problems depend on Euc. I. 37 and on the locus found in § 23, Ex. 5.

1. ABC is a triangle, and P is any point in AB; find a point Q in BC produced such that △ PBQ = △ ABC.



Construction—

Join PC.

Through A draw AQ ∥ PC, cutting BC in Q. [Euc. I. 31.

Q shall be the required point.

Proof— Join PQ.

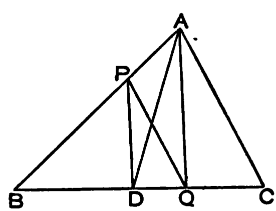
△ QPC = △ APC. [Euc. I. 37.

Add to each △ PBC;

∴ △ PBQ = △ ABC.

2. Bisect a triangle by a line drawn through a point in one of its sides.

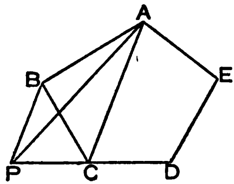
Let ABC be a triangle,
P the given point in AB.
Bisect BC in D.
Join AD.
Then △ ABD = $\frac{1}{2}$△ ABC.
[Euc. I. 38.



It is required to find a point Q in BD produced such that △ PBQ = △ ABD. This is done as in Ex. 1.

Observe that when AP > BP, Q falls beyond C, and we must then bisect AC in E, join BE, etc.

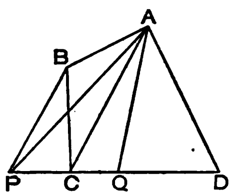
3. Construct a rectilineal figure equal to a given rectilineal figure, and having fewer sides by one than the given figure.



Let ABCDE be the figure.
Join AC.
Draw BP || AC meeting DC produced in P.
Join AP,
and prove APDE=ABCDE.

By successive constructions similar to the above a triangle may be obtained equal to a rectilineal figure of any given number of sides.

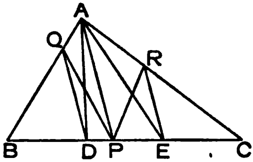
4. Bisect a quadrilateral by a straight line drawn through an angular point.



Let ABCD be the quadrilateral.
Construct, as in Ex. 3,
△ APD=ABCD.
Bisect PD in Q.
Show that AQ bisects (1) △ APD;
and (2) the quadrilateral ABCD.

Observe that when △ ABC > △ ACD, Q falls beyond C, and in that case we must produce BC and draw through D a line parallel to AC.

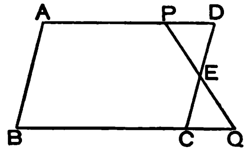
5. Trisect a triangle by straight lines drawn through a point in one of its sides.



Trisect BC in D, E,
[§ 22, Exx. 9 and 10.
and use construction of Ex. 1,
to make △ PBQ=△ ABD,
△ PRC=△ ACE.

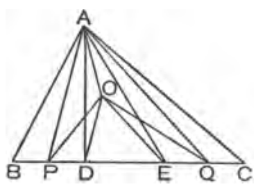
Examine the case when P is between B and D.

6. ABCD is a parallelogram, and P is a point in AD; find a point Q in BC produced such that the trapezium AQ = ▱ AC.



Bisect DC in E.
Join PE,
and produce it to meet BC in Q.
Use Euc. I. 26.

7. Trisect a triangle by straight lines drawn from a given point within the triangle.



Let BC be trisected in D, E.
Join OA, OD, OE.
Draw AP, AQ ∥ OD, OE.
Join OP, OQ.

Show that OA, OP, OQ are the lines required, using Euc. I. 38.

8. Construct a triangle equal to a given triangle having its base equal to a given straight line, and an angle equal to one of the angles of the given triangle.

Use the method of Ex. 1.

9. Construct a triangle equal to a given triangle, having its base equal to a given straight line, and having an angle equal to a given angle.

Use the method of Ex. 1, and Euc. I. 37.

10. Construct a triangle equal to a given triangle, having two of its sides equal to two given straight lines.

Use the method of Ex. 1, and Euc. I. 37.

11. Construct a triangle equal to a given triangle, having its altitude equal to a given straight line and an angle equal to one of the angles of the given triangle.

12. Construct a triangle equal to any number of given triangles.

Use the method of Ex. 11 to obtain equivalent triangles with a common altitude, and construct a triangle with this altitude and a base equal to the sum of the bases.

13. On a given base construct an isosceles triangle equal to a given triangle.

Use the methods of Ex. 1 and § 24, Ex. 9.

14. Bisect a quadrilateral by a straight line drawn through a point in one side.

Use construction of Ex. 3 to obtain an equivalent triangle with the given point as vertex.

15. Bisect a pentagon by a straight line drawn through an angular point.

Use the methods of Exx. 3 and 4.

16. Divide a triangle into four equal parts by straight lines drawn (1) through an angular point, (2) through a point in one of its sides.

See § 22, Ex. 11, and § 25, Exx. 2 and 4.

17. Divide a parallelogram into four equal parts by straight lines drawn from a point in one of its sides.

See § 22, Ex. 2, and § 25, Ex. 4.

18. Draw a straight line from an angle of a triangle to the opposite side so as to cut off a part equal to a given rectilineal figure.

19. AB is a given straight line, CDE a given triangle; find a point P in AB so that △PDE = △CDE.

Find in what case there is no solution.

§ 26. *Straight Lines which satisfy two Conditions.*

In § 24 the position of a point which satisfies two conditions is found by the method of *Intersection of Loci.* A similar method enables us to find the position of *a Straight Line which satisfies two Conditions.*

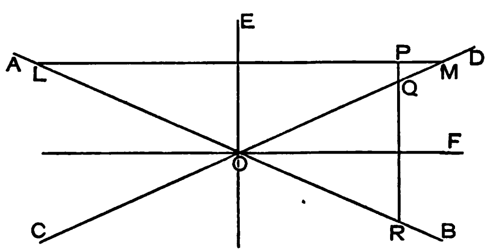
1. Draw a straight line through a given point which shall make equal angles with two given straight lines.

Here there are two conditions *to be satisfied.*

(1) *The line must pass through the given point* P. *This shows that it must be one of an infinite set of lines all radiating from* P.

(2) *The line must make equal angles with the two given straight lines. It will therefore be the base of an isosceles triangle of which the two given lines are the sides. This shows that it is one of an infinite set of lines which are all perpendicular to one and therefore parallel to the other of the bisectors of the angles formed by the given lines* (§ 1, *Ex.* 2 *and* § 4, *Ex.* 1).

The only straight lines which are common to both these sets, and thus satisfy both conditions, are found by the following construction.



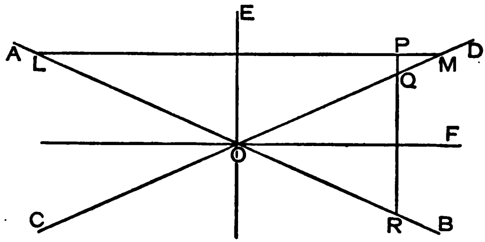
Let AOB, COD be the given straight lines, P the given point. It is required to draw a straight line through P which shall make equal angles with AB, CD.

Construction—

Draw OE, OF bisecting the angles AOD, BOD. [Euc. I. 9.

Through P draw LPM, PQR ∥ OF, OE. [Euc. I. 31.

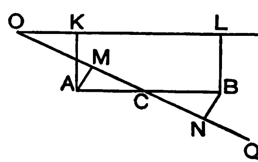
These shall be the straight lines required.



Proof—

	∠ LMO = ∠ DOF	[Euc. I. 29.
	∠ MLO = ∠ BOF	[Euc. I. 29.
and	∠ DOF = ∠ BOF ;	[Construction.
∴	∠ LMO = ∠ MLO.	
Similarly	∠ QRO = ∠ RQO.	

2. Draw a straight line through a given point O such that the perpendiculars drawn to it from two given points A, B shall be equal.



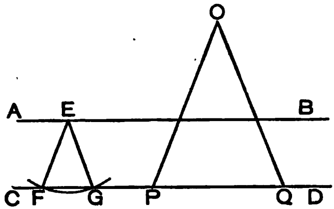
The second condition is satisfied by

(1) All straight lines ∥ AB ;

(2) All straight lines through the mid-point of AB.

Of these OP and OQ satisfy the condition of passing through O.

3. AB, CD are two parallel straight lines, and O is a given point; draw through O a straight line such that the part intercepted by AB, CD shall equal a given straight line.



With centre E on AB, and given straight line as radius, describe the circle FG.

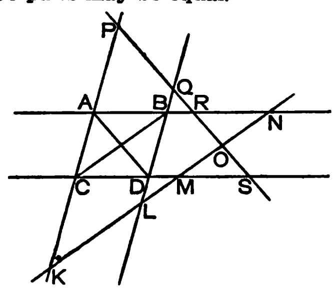
Through O draw OP, OQ || EF, EG.

4. AB, CD and AC, BD are two pairs of parallel straight lines; draw a straight line through a given point O such that the parts intercepted by the two pairs may be equal.

Join AD, BC, and draw parallels through O.

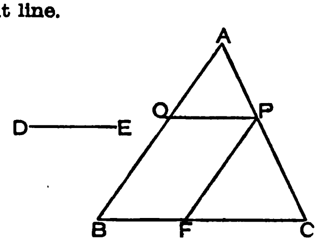
Prove by Euc. I. 34

KL=MN, PQ=RS.

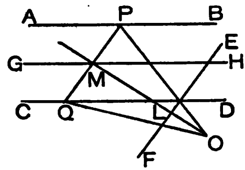


5. Draw a straight line parallel to the base of a triangle such that the part intercepted by the sides of the triangle may be equal to a given straight line.

Cut off BF=DE, and draw FP || BA, etc.

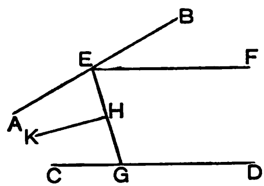


6. Draw a straight line PQ between two parallel straight lines AB, CD, parallel to a given straight line EF, and such that P and Q are equidistant from a given point O.



Draw GH equidistant from AB, CD.
Draw OL $\perp$ EF meeting GH in M.
Draw PQ through M $\parallel$ EF.
Show that PM $=$ QM and PO $=$ QO.

7. Draw a straight line which would if produced bisect the angle between two given straight lines which cannot be produced to meet.



Draw EF $\parallel$ CD, and let EG bisect $\angle$ AEF.
Bisect EG in H, and draw HK $\perp$ EG.

8. Draw a straight line of given length perpendicular to the base and terminated by one of the sides of a given triangle.

Compare Ex. 5.

DEFINITION.—*One rectilineal figure is said to be* **inscribed** *in another when each vertex of the first figure is situated on a side (or side produced) of the other.*

Observe that each vertex must be on a different side, except when (as in Ex. 9) the number of vertices exceeds the number of sides.

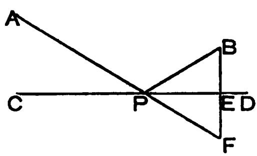
9. Inscribe a rectangle in a triangle having a side equal to a given straight line. Use Ex. 5.

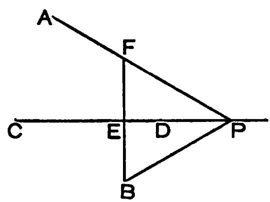
10. Inscribe a rectangle in a square having a side equal to a given straight line.

§ 27. *Analysis and Synthesis.*

Many Problems may be solved by the very important method of *Analysis and Synthesis.* This method consists in supposing for the moment that the problem has been solved, and examining or *analysing* the figure thus obtained with a view to finding a property which will lead to the required construction or *synthesis.*

1. Find a point in a given straight line such that the straight lines joining it to two given points may make equal angles with the given straight line.





Analysis—

Let A, B *be the given points,* CD *the given straight line.* *Suppose that a point* P *has been found such that*

$$\angle APC = \angle BPD.$$

It is clear that if BE *be drawn* $\perp$ CD *and produced to meet* AP *in* F, $\triangle BPE \equiv \triangle FPE$. [Euc. I. 26.

$$\therefore\ BE = FE.$$

This suggests the following

Construction—

Draw $BE \perp CD$. [Euc. I. 12.

Produce BE to F so that $EF = BE$.

Join AF, meeting CD in P. Join BP.

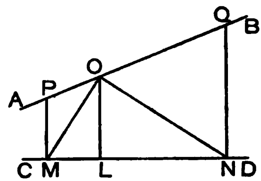
P shall be the point required.

Proof— In $\triangle$s BEP, FEP $\left\{ \begin{array}{l} BE = FE, \\ EP = EP, \\ \angle BEP = FEP; \end{array} \right.$ [Construction. [Right angles.

$\therefore \triangle BEP \equiv \triangle FEP$; [Euc. I. 4.

$\therefore \angle APC = \angle BPD.$

2. AB, CD are two given straight lines, and O is a point in AB; find points in AB equally distant from O and from CD.



Analysis—

If OP = PM, *and* PM, OL ⊥ CD,

∠ POM = ∠ PMO = ∠ MOL;

∴ OM *bisects* ∠ POL.

Construction—

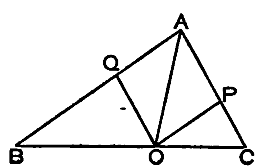
Draw OL ⊥ CD.

Draw OM, ON bisecting ∠s AOL, BOL.

Draw MP, NQ ⊥ CD.

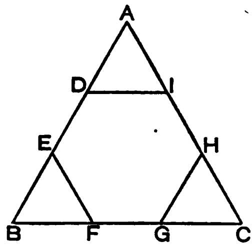
P and Q shall be the points required.

3. Find a point in the base of a triangle such that straight lines drawn from that point parallel to and terminated by the sides shall be equal.



APOQ is a rhombus, and we know that the diagonal of a rhombus bisects its angles. Hence we obtain a construction to find O.

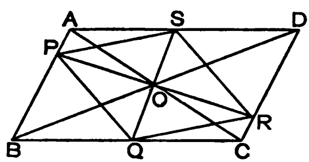
4. Inscribe a regular hexagon in an equilateral triangle.



It is clear that each side must be trisected.

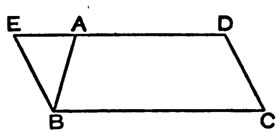
5. Inscribe a rhombus in a parallelogram with a given point in one side as vertex.

Let P be the given point.
Draw POR through the centre O.
Draw QOS ⊥ POR.
Prove PQRS to be the required rhombus.



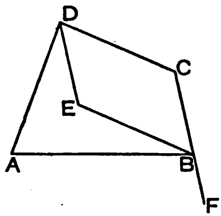
Examine the cases in which either two or four of the vertices of the rhombus are on the produced sides.

6. Construct a trapezium whose sides are given.



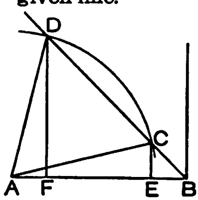
Construct the △EAB so that EB, BA are the oblique sides and EA the difference between the parallel sides.
Complete the ▱ EDCB.
ABCD is the trapezium required.

7. Construct a quadrilateral having given two opposite sides and three angles.



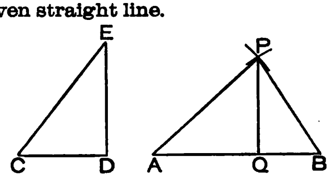
Let AB, CD = the given sides,
and $\angle$s A, B, C = the given angles.
Draw AB, make $\angle$ BAD = $\angle$ A, and $\angle$ ABC = $\angle$ B.
Produce CB to F,
and make $\angle$ FBE = $\angle$ C.
Cut off BE = CD.
Draw ED $\parallel$ BC meeting AD in D.
Draw DC $\parallel$ EB meeting BC in C.
Prove ABCD to be the required quadrilateral.

8. Divide a given straight line into two parts, so that the sum of the squares on the segments may be equal to the square on a given line.



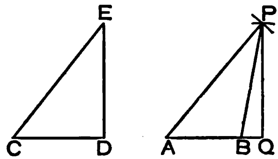
Let AB be the given line to be divided,
AC^2 the given square.
If $\angle$ ABC $= \frac{1}{2}$ rt. $\angle$, EC = EB,
$AE^2 + EB^2 = AE^2 + EC^2 = AC^2$.
Also $AF^2 + FB^2 = AF^2 + FD^2 = AD^2$.
Hence we obtain the construction indicated in the figure.

9. Divide a given straight line into two parts so that the difference of the squares on the segments may be equal to the square on a given straight line.



Let AB be the given straight line, CD^2 the given square.
Suppose Q to be the required point and QP $\perp$ AB.
Then $AP^2 - BP^2 = AQ^2 - BQ^2 = CD^2 = CE^2 - ED^2$,
if DE $\perp$ CD.
$\therefore$ on base CD construct right-angled $\triangle$ CDE,
and make AP = CE, BP = DE, etc.

10. Produce a given straight line, so that the difference of the squares on the whole line thus produced, and on the part produced, may be equal to the square on a given straight line.



Let AB be the given straight line, CD^2 the given square.

Suppose Q to be the required point and $QP \perp AB$.

Then $AQ^2 - BQ^2 = AP^2 - BP^2 = CD^2 = CE^2 - ED^2$,

if $DE \perp CD$.

Hence derive a construction to obtain $\triangle$ APB and to find Q.

11. Divide a straight line AB in Q so that $AQ^2 = 2BQ^2$.

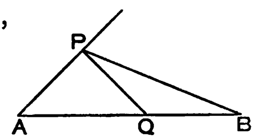
If APQ be a rt. $\angle$, and $AP = PQ = QB$,

then $AQ^2 = 2QB^2$.

Hence we see that $\angle PAQ = \frac{1}{2}$ rt. $\angle$;

and $\angle PBQ = \frac{1}{2} \angle AQP = \frac{1}{4}$ rt. $\angle$.

This leads to a construction to find P.



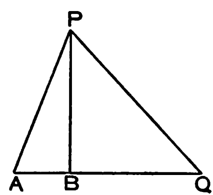
12. Produce a straight line AB to Q so that $AQ^2 = 2BQ^2$.

If $AQ^2 = PQ^2 = PB^2 + BQ^2 = 2BQ^2$,

we see that $\angle Q = \frac{1}{2}$ rt. $\angle$;

$\therefore \angle A = \frac{3}{4}$ rt. $\angle$. [Euc. I. 32.

Thus we obtain BP, and therefore BQ.

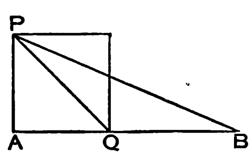


13. Construct a square, having given the sum of a side and a diagonal.

Given AB, make $\angle ABP = \frac{1}{4}$ rt. $\angle$.

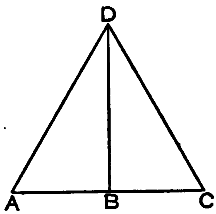
Draw $AP \perp AB$, and take $AQ = AP$.

(AB may also be divided as in Ex. 11.)



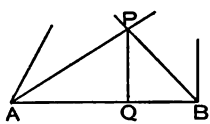
14. Draw a straight line BD such that $BD^2=3AB^2$.

This problem may be solved as in § 22, Exx. 6 and 21, or by the following method :—



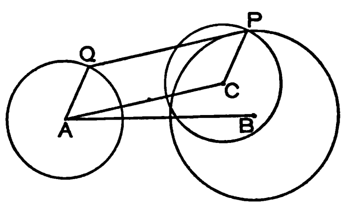
Draw $BD \perp AB$. If $BD^2=3AB^2$,
$AD^2=AB^2+BD^2=4AB^2$;
$\therefore AD=2AB$.
$\therefore$ if AB be produced to C so that $BC=AB$, and CD be joined, ACD is an equilateral triangle.
Hence obtain a construction.

15. Divide a straight line AB in Q so that $AQ^2=3BQ^2$.



If $PQ=QB$, $AP^2=4PQ^2$.
Thus $\angle PAQ$=half the angle of an equilateral triangle,
and $\angle PBQ$=half a right angle.
Hence obtain a construction.

16. Having given two circles, draw a line equal and parallel to a given straight line with one of its extremities on each of the circles.



Let A and B be centres of circles.
AC=and ∥ given line.
With centre C describe a circle=circle A and cutting circle B in P.

Draw AQ ∥ CP, and show that QP is the required line.

Show that there are in general two solutions, and find in what case there is no solution.

17. Inscribe a rhombus in a given triangle so that one angle of the rhombus may coincide with one angle of the triangle.

Compare Ex. 3.

18. Inscribe a square in a right-angled triangle so that one angle of the square may coincide with the right angle.

Use the method of Ex. 3.

19. Inscribe a square in a given rhombus.

Draw the diagonals and compare previous Ex.

20. Divide a straight line AB in C so that $AC^2=4BC^2$.

Trisect the line. See § 22, Ex. 9.

21. Divide AB in C so that $AC^2=\frac{1}{2}AB^2$.

See § 20, Ex. 2.

22. Produce AB to C so that $AC^2=2AB^2$.

See § 20, Ex. 2.

23. Construct a square, having given the difference between a side and a diagonal.

Compare Ex. 12.

24. Produce a straight line AB to C so that $AC^2=3BC^2$.

Use the method of Ex. 15, but bisect the exterior right angle at B.

25. Find two straight lines, having given the sum of the lines and the difference of their squares.

Compare Ex. 9.

26. Find two straight lines, having given the difference of the lines and the difference of their squares.

Compare Ex. 10.

27. Given two circles and a point, draw a line through the point to meet the two circles and be bisected at the point.

Use § 15, Ex. 1, and a construction similar to that of Ex. 16.

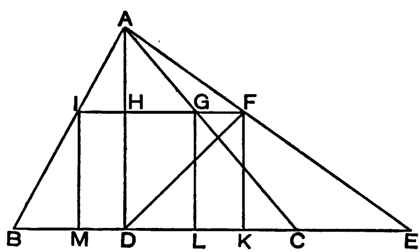
Find the condition that this problem may have two, one, or no solutions.

§ 28. *Reduction to a Simpler Case.*

Many problems may be solved by the method of *Reduction to a Simpler Case.*

1. To inscribe a square in a given triangle.

In § 27, *Ex.* 3, *a rhombus has been inscribed in a triangle. If the triangle be right-angled the rhombus will be a square. We thus see how to inscribe a square in a right-angled triangle.*



Let ABC be a triangle; it is required to inscribe a square in ABC.

Construction—

Draw AD ⊥ BC. [Euc. I. 12.

Produce BC to E, so that CE = BD.

Bisect ∠ ADE by DF, meeting AE in F. [Euc. I. 9.

Through F draw FI ∥ CB, [Euc. I. 31.

and let FI meet AC, AD, AB in G, H, I.

Draw FK, GL, IM ⊥ BC. [Euc. I. 12.

IMLG shall be the inscribed square.

Proof—

The figures HDKF, IMLG are rectangles. [Construction.

also $\angle$ HDF = $\angle$ KDF, [Construction.

and $\angle$ KDF = $\angle$ HFD; [Euc. I. 29.

$\therefore$ $\angle$ HDF = $\angle$ HFD;

$\therefore$ HF = HD; [Euc. I. 6.

$\therefore$ HDKF is a square.

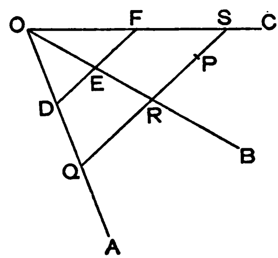
And the intercepts, made by the sides of two triangles, which have equal bases in the same straight line and are between the same parallels, on a straight line parallel to their bases, are equal;

$\therefore$ IH = GF; [§ 18, Ex. 3.

$\therefore$ IG = HF = HD = IM;

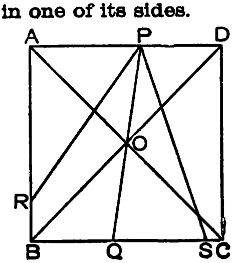
or IMLG is a square, and it is inscribed in the $\triangle$ ABC.

2. Draw a straight line through a given point such that the intercepts made on it by three concurrent straight lines shall be equal.



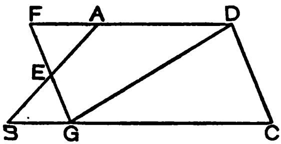
Use the construction of § 22, Ex. 4, to draw DEF so that DE = EF. Through P draw QRPS ∥ DF. Then QR = RS. [§ 18, Ex. 2.

3. Divide a square (or a parallelogram) into four equal parts by straight lines drawn through a point in one of its sides.



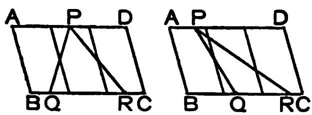
Use the methods of § 22, Ex. 2, and § 25, Ex. 4.

4. Bisect a trapezium by a straight line drawn through one of its angular points.



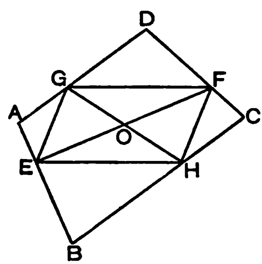
Let ABCD be the trapezium.
Use the construction of § 17, Ex. 14, and draw DG.

5. Trisect a parallelogram by straight lines drawn through a point in one of its sides.



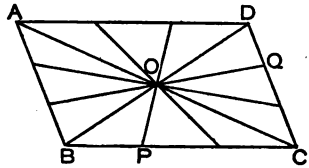
First, trisect the parallelogram as in § 22, Ex. 23; then use the methods of § 25, Ex. 6, and § 25, Ex. 4.

6. Any two points being given in the opposite sides of a quadrilateral, inscribe in it a parallelogram having two of its angles at the two given points.



Let ABCD be quadrilateral; E, F the given points.
Bisect EF in O, and use the construction of § 24, Ex. 7.
Draw GOH so that GH is bisected in O.
Use § 15, Ex. 1.
Examine the cases in which one or more of the vertices of the parallelogram are on produced sides.

7. Trisect a parallelogram by straight lines drawn from its centre.



The diagonals divide the parallelogram into four equal triangles. If now each of the sides be trisected and the centre O be joined to the points of trisection, the parallelogram is divided into twelve equal triangles, of which each of the required parts must contain four.

Thus OA, OP, OQ form a solution.

8. Cut off from a parallelogram any part required (say two-fifths) by a straight line drawn parallel to one of its sides.

Use the methods of § 22, Exx. 10, 23.

9. Cut off from a parallelogram any part required (say two-fifths) by a straight line drawn through a point in one of its sides.

Use the method of Ex. 5.

10. Cut off from a triangle any part required by a straight line drawn through one of its angular points.

See § 22, Ex. 11.

11. Cut off from a triangle any part required by a straight line drawn through a point in one of its sides.

Compare the previous Ex. and § 25, Ex. 1.

12. Cut off from a quadrilateral any part required by a straight line drawn through an angular point.

Use a method similar to § 25, Ex. 4.

13. Divide a parallelogram into five equal parts by lines drawn through its centre.

Use the method of Ex. 7.

14. Inscribe a parallelogram in a given triangle, so that its diagonals may intersect at a given point within the triangle.

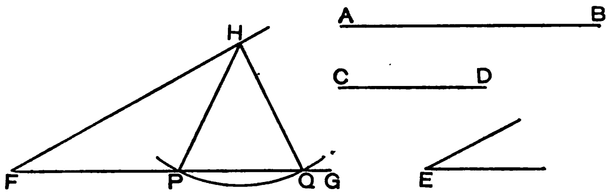
Use the method of § 24, Ex. 7, and § 15, Ex. 1.

§ 29. *Construction of Triangles.*

The construction of triangles, when certain parts or certain relations between the parts are given, gives rise to many problems.

Two problems of this kind are solved in Euc. I. 1 and 22, and another is given in § 24, Ex. 3. When two sides and the contained angle, or two $\angle$s and the adjacent side of a triangle, are given, the solutions are obvious.

1. Construct a triangle, having given two sides and an angle opposite to one of the given sides.



Let AB, CD be the given sides; E the given angle which is to be opposite CD.

Construction—

Draw any straight line FG.

At F make $\angle$ GFH $=$ $\angle$ E. [Euc. I. 23.

Cut off FH $=$ AB.

With H as centre and radius $=$ CD,

describe a circle cutting FG in P, Q.

Then either FHP or FHQ shall be the triangle required.

Proof—

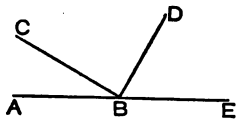
FH $=$ AB, [Construction.

HP $=$ HQ $=$ CD, ,,

$\angle$ HFP $=$ $\angle$ E. ,,

Observe that if this circle does not cut FG there is no solution. If CD $>$ AB, the point P lies in GF produced beyond F, and there is only one solution. Compare § 10, Ex. 1.

2. Construct a triangle, having given two angles and a side opposite to one of them.



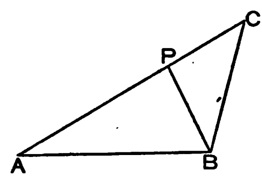
Let ∠s ABC, CBD be the given angles.

Produce AB to E. Then ∠ DBE is the third angle of the triangle. [Euc. I. 32.

Thus we know two angles and the side adjacent to both of them.

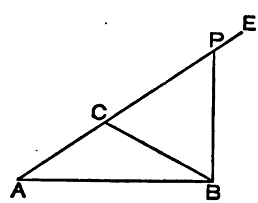
3. Construct a triangle, having given the base, an angle adjacent to the base, and the sum of the sides.

AB = base, ∠ BAC = given angle,
AC = sum of sides.
Join BC, and make ∠ CBP = ∠ ACB.
ABP is triangle required.

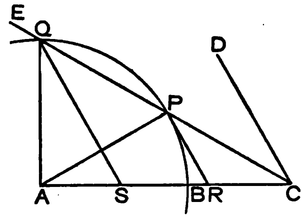


4. Construct a triangle, having given the base, an angle adjacent to the base, and the difference of the sides.

AB = base, ∠ BAC = given angle,
AC = difference of sides.
Join BC. Produce AC to E.
Make ∠ CBP = ∠ BCE.
ABP is triangle required.

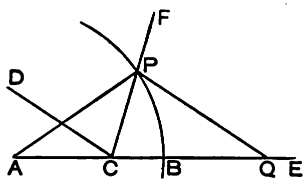


5. Construct a triangle, having given the base, the vertical angle, and the sum of the sides.



Let AB = base, AC = sum of sides,
$\angle$ ACD = vertical angle.
Draw CE bisecting $\angle$ ACD.
With centre A and radius AB, describe a circle cutting CE in P, Q. Draw PR, QS || DC.
Both $\triangle$s APR, QAS satisfy the conditions.
They may be shown to be congruent.

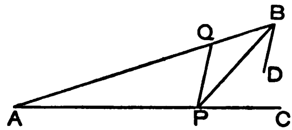
6. Construct a triangle, having given the base, the vertical angle, and the difference of the sides.



Let AB = base, AC = difference of sides,
$\angle$ ACD = vertical angle.
Produce AC to E, and let CF bisect $\angle$ DCE.
With centre A, and radius AB, describe a circle cutting CF in P.
Draw PQ || DC, cutting AE in Q.
APQ is triangle required.

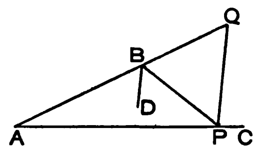
7. Construct a triangle, having given the sum of the sides, the vertical angle, and one of the angles at the base.

AB = sum of sides,
∠ ABD = vertical angle,
∠ BAC = angle at base.
BP bisects ∠ ABD, etc.
APQ is triangle required.

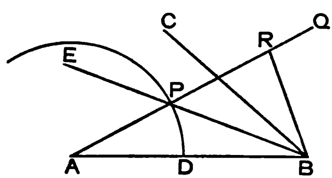


8. Construct a triangle, having given the difference of the sides, the vertical angle, and one of the angles at the base.

AB = difference of sides,
∠ ABD = vertical angle,
∠ BAC = angle at base.
BP bisects ∠ QBD, etc.
APQ is triangle required.

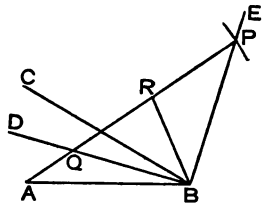


9. Construct a triangle, having given the base, the difference of the angles at the base, and the difference of the sides.



AB = base, ∠ ABC = difference of angles at base,
AD = difference of sides.
BPE bisects ∠ ABC.
AP is produced to Q and ∠ CBR made = ∠ PAB.
ABR is triangle required.

10. Construct a triangle, having given the base, the difference of the angles at the base, and the sum of the sides.



AB = base, ∠ ABC = difference of angles.

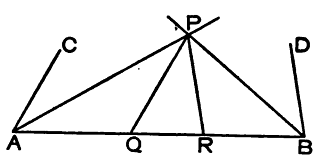
BD bisects ∠ ABC. BE ⊥ BD. AP = sum of sides.

AP meets BD in Q. R is mid-point of PQ.

ABR is triangle required.

Observe that QRB is an isosceles triangle. [§ 12, Ex. 1.

11. Construct a triangle, having given the perimeter and two angles.



AB = perimeter.

∠ BAC, ∠ ABD = given angles.

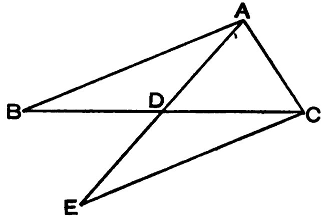
AP, BP bisect angles BAC, ABD.

PQ ∥ CA, PR ∥ DB.

PQR is triangle required.

12. Construct a triangle, having given two sides and the included median.

First construct △ ACE, with the two given sides and twice the given median as its sides. Bisect AE in D, and produce CD its own length to B.



ABC is the required triangle.

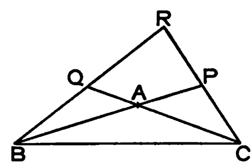
13. Construct a triangle, having given two sides and the median to one of the given sides.

Use the figure of Ex. 12. Construct △ ADC with half the side to which the median is drawn, the other given side and the given median as its sides.

Hence obtain the required triangle.

14. Construct a triangle, having given one side and the medians drawn from its extremities.

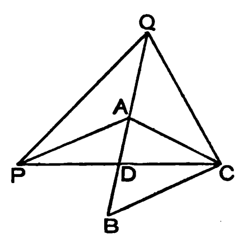
First construct △ ABC, with the given side BC, and $\frac{2}{3}$ of each of the given medians as its sides. Produce BA and CA half their lengths, etc.



RBC is triangle required.

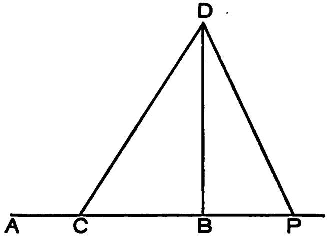
15. Construct a triangle having given the three medians.

First construct △ ABC with $\frac{2}{3}$ of each of the given medians as its sides. Bisect AB in D, and produce CD its own length to P, and BA its own length to Q.



QPC is the required triangle.

16. Construct a right-angled triangle, having given one side and the difference between the hypotenuse and the sum of the two sides which contain the right angle.



AB = given side,

AC = given difference.

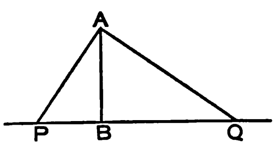
Draw BD ⊥ AB and = AB.

Join CD.

Make ∠ CDP = ∠ DCB.

DBP is the required triangle.

17. Construct a right-angled triangle, having given the perpendicular from the right angle on the hypotenuse, and one of the acute angles.



AB = given perpendicular,

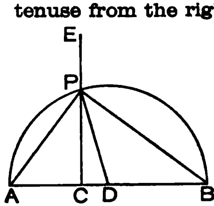
PBQ ⊥ AB;

∠ BAP = given angle;

and AQ ⊥ AP.

APQ is the required triangle.

18. Construct a right-angled triangle, having given the hypotenuse and the foot of the perpendicular on the hypotenuse from the right angle.

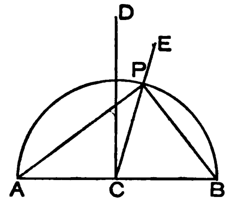


AB = hypotenuse.

CE = perpendicular.

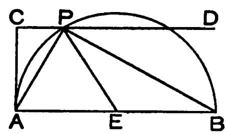
Use property of § 12, Ex. 1, to find P.

19. Construct a right-angled triangle, having given the hypotenuse and the difference of the base angles.



Bisect AB in C, draw CD ⊥ AB, and make ∠ DCE = given difference. From CE cut off CP = AC. APB is required triangle. Compare § 12, Ex. 5.

20. Construct a right-angled triangle, having given the hypotenuse and the perpendicular on the hypotenuse from the right angle.



AB = hypotenuse,
AC = altitude.
E is mid-point of AB.
EP = EA; hence construction.

21. Construct a triangle, having given the mid-points of the sides. Compare § 16, Ex. 2.

22. Construct a right-angled triangle, having given the hypotenuse and one side.

Use the method of Ex. 1.

23. Construct a right-angled triangle, having given the hypotenuse and the sum of the sides.

Use the method of Ex. 5.

24. Construct a right-angled triangle, having given the hypotenuse and the difference of the sides.

Use the method of Ex. 6.

25. Construct a right-angled triangle, having given the perimeter and one of the acute angles.

Use the method of Ex. 11.

26. Construct an isosceles right-angled triangle, having given the sum of the hypotenuse and one side.

Use the method of § 27, Ex. 11.

27. Construct an isosceles right-angled triangle, having given the difference of the hypotenuse and one side.

Use the method of § 27, Ex. 12.

28. Construct a parallelogram, having given two adjacent sides and one diagonal.

Use Euc. I. 22.

29. Construct a parallelogram, having given one side and both diagonals.

Use Euc. I. 22, bisecting both diagonals.

30. Construct a right-angled triangle, having given one side and the sum of the hypotenuse and the other side.

Use the method of Ex. 3.

31. Construct an isosceles triangle, having given the base, and the sum of one of the equal sides and the perpendicular from the vertex to the base.

Use previous Ex. to construct one of the two equal right-angled triangles into which the perpendicular divides the triangle.

32. Construct a triangle, having given an angle, an altitude drawn from another angle, and the perimeter.

This reduces to the case of Ex. 3.

33. Construct an isosceles triangle, having given the base and the difference between one of the equal sides and the perpendicular from the vertex to the base.

Use the method of Ex. 4 to construct half the triangle.

34. Construct an isosceles triangle, having given the perimeter and the altitude from the vertical angle.

See Ex. 30.

35. Construct an isosceles triangle, having given the vertical angle and the altitude from it.

36. Construct a right-angled triangle, having given an acute angle and the sum of the sides which contain the right angle.

Use the method of Ex. 7.

37. Construct a right-angled triangle, having given an acute angle and the difference of the sides which contain the right angle.

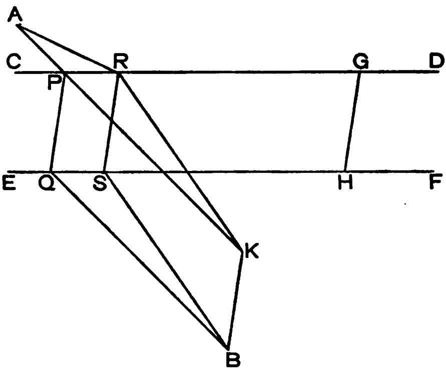
Use the method of Ex. 8.

38. Find two straight lines, having given their difference and the sum of their squares.

Compare Ex. 24.

§ 30. *Miscellaneous Problems.*

1. A and B are two points outside two parallel straight lines, CD, EF; find two points, P, Q in CD, EF, such that PQ shall be parallel to a given straight line GH, and that the rectilineal path APQB shall be the shortest possible.



Construction—

From B draw BK= and ∥ HG. [Euc. I. 31.

Join AK meeting CD in P.

Draw PQ ∥ GH, meeting EF in Q. [„

Join BQ. APQB shall be the path required.

Proof—

Draw any other straight line RS ∥ GH, meeting CD, EF in R, S.

Join AR, BS.

PQBK, RSBK are both parallelograms;

$\therefore$ AP+PQ+QB=AK+KB. [Euc. I. 34.

Also AR+RS+SB=AR+RK+KB; [„

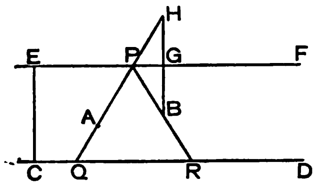
but AR+RK>AK; [Euc. I. 20.

$\therefore$ AR+RS+SB>AP+PQ+QB.

But ARSB is any other path subject to the given condition;

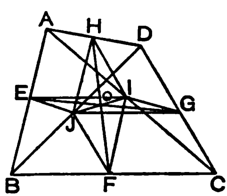
$\therefore$ APQB is the shortest possible path.

2. Construct an isosceles triangle of given altitude, whose sides shall pass through two given points, and which shall have its base on a given line.



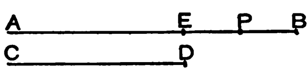
Let A, B be the given points;
CD the given line;
CE the given altitude.
Draw EF $\parallel$ CD, and use the construction of § 27, Ex. 1.

3. Construct a quadrilateral, having given its four sides and the line ioining the mid-points of two opposite sides.



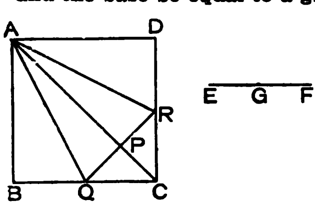
Let EG be the given line.
Since $EJ = \frac{1}{2}AD$, $JG = \frac{1}{2}BC$.
We can construct the ▱EIGJ;
thus IJ is known.
Similarly, we can construct ▱FIHJ.
The $\triangle$s AEH, BEF, etc., can now be constructed.

4. Find two straight lines, having given their sum and difference.



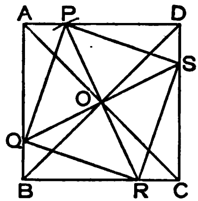
AB = sum:
AE = CD = difference:
P is mid-point of EB.
AP, PB are the required lines.

5. Inscribe an isosceles triangle in a given square, so that the vertex may coincide with an angular point of the square and the base be equal to a given straight line.



Let AQR be the triangle.
AC bisects QR in P.
By § 12, Ex. 1,
$PC = \frac{1}{2}QR = \frac{1}{2}EF$.
Hence obtain a construction.

6. Inscribe a square in a given square, having its diagonal equal to a given straight line.

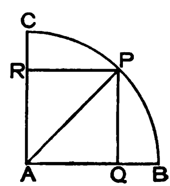


With O as centre and radius $= \frac{1}{2}$ given diagonal,
describe a circle cutting AD in P.
Join OP, and produce PO to meet BC in R.
Draw QOS $\perp$ PR, etc.

Consider the case in which P lies on AD produced.

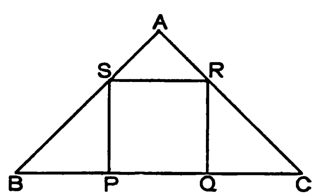
7. Inscribe a square in a quadrant of a circle.

Bisect $\angle$ BAC, etc.

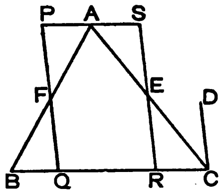


8. Inscribe a square in a right-angled isosceles triangle, one side being on the hypotenuse.

Trisect BC in P, Q, etc.



9. Divide a triangle by two straight lines into three parts which can be arranged so as to form a parallelogram with a given angle.



Bisect AB, AC in F, E.

Let DCB = given angle, etc.

10. A, B are two points on the same side of a straight line CD; find a point P in CD such that AP+BP may be a minimum.

Use § 21, Ex. 1, and compare § 27, Ex. 1.

11. Inscribe an equilateral triangle in a square, so that a vertex of the triangle may coincide with an angular point of the square.

Make an angle with the diagonal equal to half the angle of an equilateral triangle.

12. Inscribe an equilateral triangle in a square, so that a vertex of the triangle may coincide with the mid-point of a side of the square.

The sides of the triangle are equal to the sides of the square.

13. In a given square inscribe a square having its sides equal to a given straight line.

Find diagonal of required square, and see Ex. 6.

14. Given the base and area of a triangle, find its vertex so that its perimeter may be a minimum.

See § 21, Ex. 3.

15. Given the base and perimeter of a triangle, find its vertex so that its area may be a maximum.

See § 21, Ex. 8.

APPENDIX.

Enunciations of the Propositions and Corollaries of EUCLID, BOOK I.

1. To construct an equilateral triangle on a given straight line.

2. From a given point to draw a straight line equal to a given straight line.

3. From the greater of two given straight lines to cut off a part equal to the less.

4. If two sides and the contained angle of one triangle be respectively equal to two sides and the contained angle of another triangle, the two triangles shall be equal in every respect.

5. The angles at the base of an isosceles triangle are equal; and, if the equal sides be produced, the angles on the other side of the base shall also be equal.

Cor.—Every equilateral triangle is also equiangular.

6. If two angles of a triangle be equal, the sides opposite to them shall also be equal.

Cor.—Every equiangular triangle is also equilateral.

7. On the same base, and on the same side of it, there cannot be two triangles having the sides which are terminated at one end of the base equal and also those which are terminated at the other end.

8. If three sides of one triangle be respectively equal to three sides of another triangle, the two triangles shall be equal in every respect.

9. To bisect a given angle.

10. To bisect a given straight line.

11. To draw a straight line at right angles to a given straight line from a given point in the line.

12. To draw a straight line perpendicular to a given straight line from a given point outside the line.

13. The angles which one straight line makes with another on one side of it are together equal to two right angles.

Cor. i.—If two straight lines cut one another, the angles formed shall be together equal to four right angles.

Cor. ii.—If any number of straight lines meet in a point, the angles formed shall be together equal to four right angles.

14. If at a point in a straight line two other straight lines on opposite sides of it make adjacent angles which are together equal to two right angles, these two straight lines shall be in one and the same straight line.

15. If two straight lines cut one another, the vertically opposite angles shall be equal.

16. If one side of a triangle be produced, the exterior angle thus formed shall be greater than either of the two interior opposite angles.

17. Any two angles of a triangle are together less than two right angles.

18. If one side of a triangle be greater than another side, the angle opposite to the greater side shall be greater than the angle opposite to the other.

19. If one angle of a triangle be greater than another, the side opposite to the greater angle shall be greater than the side opposite to the other.

20. Any two sides of a triangle are together greater than the third side.

21. If from the ends of a side of a triangle two straight lines be drawn to a point within the triangle, these two straight lines shall be together less than the other two sides of the triangle, but they shall contain a greater angle.

22. To construct a triangle whose sides shall be equal to three given straight lines, any two of which are greater than the third.

23. At a given point in a given straight line, to make an angle equal to a given angle.

24. If two triangles have two sides of the one respectively equal to two sides of the other, but the contained angles unequal, the base of the triangle which has the greater contained angle shall be greater than the base of the other.

25. If two triangles have two sides of the one respectively equal to two sides of the other but the bases unequal, the angle contained by the two sides of the triangle which has the greater base shall be greater than the angle contained by the two sides of the other.

26. If two angles and a side in one triangle be respectively equal to two angles and the corresponding side in another, the two triangles shall be equal in every respect.

27. If a straight line cutting two other straight lines make the alternate angles equal, the two straight lines shall be parallel.

28. If a straight line cutting two other straight lines make an exterior angle equal to the interior opposite angle on the same side of the cutting line, or make the two interior angles on the same side of the cutting line together equal to two right angles, the two straight lines shall be parallel.

29. If a straight line cut two parallel straight lines, it shall make the alternate angles equal, and any exterior angle equal to the interior opposite angle on the same side of the cutting line, and the two interior angles on the same side of the cutting line together equal to two right angles.

30. Straight lines which are parallel to the same straight line are parallel to each other.

31. Through a given point, to draw a straight line parallel to a given straight line.

32. If one side of a triangle be produced, the exterior angle shall be equal to the sum of the two interior opposite angles, and the three interior angles are together equal to two right angles.

Cor. i.—All the interior angles of any rectilineal figure together with four right angles are equal to twice as many right angles as the figure has sides.

Cor. ii.—All the exterior angles of any rectilineal figure are together equal to four right angles.

33. The straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

34. The opposite sides and opposite angles of a parallelogram are equal, and either diagonal bisects the parallelogram.

35. Parallelograms on the same base and between the same parallels are equal.

36. Parallelograms on equal bases and between the same parallels are equal.

37. Triangles on the same base and between the same parallels are equal.

38. Triangles on equal bases and between the same parallels are equal.

39. Equal triangles on the same side of the same base are between the same parallels.

40. Equal triangles on the same side of equal bases which are in the same straight line are between the same parallels.

41. If a parallelogram and a triangle be on the same base and between the same parallels, the parallelogram shall be double the triangle.

42. To construct a parallelogram which shall be equal to a given triangle and have an angle equal to a given angle.

43. The complements of the parallelograms which are about a diagonal of a parallelogram are equal.

44. On a given straight line to construct a parallelogram which shall be equal to a given triangle and have an angle equal to a given angle.

45. To construct a parallelogram which shall be equal to a given rectilineal figure and have an angle equal to a given angle.

46. To construct a square on a given straight line.

47. In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

48. If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by these two sides shall be a right angle.

STANDARD THEOREMS.—See Geometrical Deductions, § 1, Exx. 2 and 4; § 3, Exx. 1 and 6; § 4, Ex. 1; § 5, Ex. 1; § 8, Ex. 1; § 9, Exx. 1, 6, and 7; § 10, Ex. 1; § 12, Ex. 1; § 15, Ex. 1; § 16, Exx. 1, 2, 3, 8, and 9; § 17, Exx. 1 and 4; § 18, Exx. 1, 2, 3, 5, and Corollaries; § 21, Ex. 1.

STANDARD LOCI.—See § 23, Exx. 1, 2, 3, 4, and 5.

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